Exercises

Exercise 1. Proof Lemma 1.1: For all $x, y \in V$ we have $x \wedge y = -y \wedge x$.

Exercise 2. Check that the vectors $v_1, \ldots, v_k$ are linearly dependent if and only if $v_1 \wedge \cdots \wedge v_k = 0$.

Exercise 3. Prove Lemma 1.4: Let $x = v_1 \wedge \cdots \wedge v_k \in \Lambda^k(V)$ be decomposable, M as above. Then we have

$$p_{i_1,\ldots,i_k}(x) = \triangle^{1,2,\ldots,k}_{i_1,\ldots,i_k}(M).$$

Hint: use the expansion of the v_i in terms of $e_1, \ldots, e_n$.

Exercise 4. Write the Plücker relations for $\mathrm{Gr}(2,5)$.

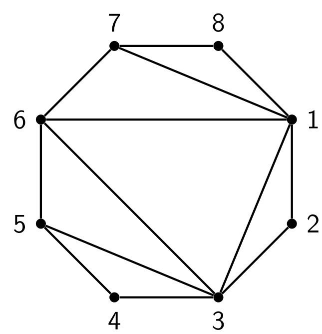
Exercise 5. Check the following: For $k = 2$, the Plücker relations are

$$p_{i,j_0}p_{j_1,j_2} - p_{i,j_1}p_{j_0,j_2} + p_{i,j_2}p_{j_0.j_1}$$

where $1 \leq i \leq n$, $1 \leq j_0 < j_1 < j_2 \leq n$. We can rewrite these as

$$p_{ab}p_{cd} - p_{ac}p_{bd} + p_{ad}p_{bc} \quad \text{for all } a,b,c,d \text{ with } 1 \leq a < b < c < d \leq n$$

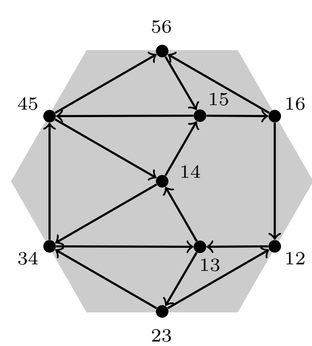
Exercise 6. Find Q_T for the triangulation T given by the diagonals $(13), (35), (36), (16), (17)$ of an octagon.



Exercise 7. Draw the $\sigma_{2,8}$-diagram D for the triangulation T from Exercise 6. Compare the two quivers Q_T and $Q(D)$.

Exercise 8. Find a triangulation T such that $Q(T)$ as defined in Example 2.3(b) is the dimer of Example 2.2.

Exercise 9. Any two unit cycles at a vertex of a dimer model Q commute. Why is this?



Exercise 10. Find the Postnikov diagram for the dimer model above. Determine its permutation.

Exercise 11. Show that $\mathrm{Hom}_M(\mathbb{M}_I, \mathbb{M}_J) \cong \mathbb{C}[|t|]$ for all I, J.

Example Let $n = 6$, $k = 3$. So $J_{3,6}$ is as follows:

$$\begin{array}{ccccccccc} & & & & \beta & & & & \\ & & & & | & & & & \\ \alpha_1 & — & \alpha_2 & — & \alpha_3 & — & \alpha_4 & — & \alpha_5 \end{array}$$

Let $I = \{1, 3, 5\}$ and $J = \{2, 4, 6\}$.

1. $M := \mathbb{M}_I$: $\underline{a} = (1, 0, 1, 0, 1, 0)$ and $\varphi(M) = \beta + \alpha_2 + \alpha_3 + \alpha_4$.
2. Let $M = \mathbb{M}_I \,/\, \mathbb{M}_J$. Then $\underline{a} = (1, 1, 1, 1, 1, 1)$ which corresponds to $2\beta+\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4+\alpha_5$, the highest root for E_6.

Exercise 12. Compute $\varphi(M)$ for $M = \mathbb{M}_J$ and for $M = \mathbb{M}_J \,/\, \mathbb{M}_I$ where $\mathbb{M}_I$ and $\mathbb{M}_J$ are as in the example above.

Exercise 13. Find $\varphi(\mathbb{M}_I \,/\, \mathbb{M}_J)$ for I, J as in Example 3.21 (or 3.20). Compute $q(\underline{a}(\mathbb{M}_I \,/\, \mathbb{M}_J))$.

Define matrices

$$A_1 := \begin{pmatrix} t & -2 \\ 0 & 1 \end{pmatrix} \quad B_1 := \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \quad C_1 := \begin{pmatrix} t & -1 \\ 0 & 1 \end{pmatrix} \quad D_1 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$A_2 := \begin{pmatrix} 1 & 2 \\ 0 & t \end{pmatrix} \quad B_2 := \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \quad C_2 := \begin{pmatrix} 1 & 1 \\ 0 & t \end{pmatrix} \quad D_2 := \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}.$$

Definition 3.24 (or 3.23)
Let I, J be strictly 3-interlacing k-subsets of $[n]$. At the vertices of Γ_n, $\mathbb{M}(I,J)$ has the $V_1, \ldots, V_n$. We define the maps x_i, y_i as follows:

$$x_i : V_{i-1} \to V_i \text{ acts as } \begin{cases} A_1 & \text{if } i = a_1 \\ B_2 & \text{if } i = b_1 \\ B_1 & \text{if } i = a_2 \\ C_2 & \text{if } i = b_2 \\ C_1 & \text{if } i = a_3 \\ A_2 & \text{if } i = b_3 \\ D_1 & \text{if } i \in I \cap J \\ D_2 & \text{if } i \in I^c \cap J^c \end{cases} \qquad y_i : V_i \to V_{i-1} \text{ acts as } \begin{cases} A_2 & \text{if } i = a_1 \\ B_1 & \text{if } i = b_1 \\ B_2 & \text{if } i = a_2 \\ C_1 & \text{if } i = b_2 \\ C_2 & \text{if } i = a_3 \\ A_1 & \text{if } i = b_3 \\ D_2 & \text{if } i \in I \cap J \\ D_1 & \text{if } i \in I^c \cap J^c \end{cases}$$

Exercise 14. Check that the module $\mathbb{M}(I,J)$ from the above definition is in $\mathcal{F}_{k,n}$. For this, check that $xy = yx$ and $x^k = y^{n-k}$ at all vertices (hence is a B-module) and that $\mathbb{M}(I,J)$ is free over the centre.