# Cluster algebras and cluster categories via surfaces 

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#### Abstract

Twenty years of research in cluster theory have established deep links between cluster algebras, surface geometry and representation theory. Cluster structures can be defined using surface geometry, with curves corresponding to cluster variables and to rigid objects in associated categories. The classical Ptolemy relations give rise to exchange phenomena or mutation in the associated cluster structures. In these lectures I will focus on the geometric approach to cluster algebras and cluster categories. Topics discussed will include cluster structures on the Grassmannian, friezes, laminations, Postnikov diagrams, dimer models, root combinatorics.


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## 1 Grassmannians

The main reference for this is Section 9 in B. Marsh's book on cluster algebras, [Mar13].

### 1.1 Exterior powers

Let $V:=\mathbb{C}^{n}$, the tensor algebra is defined as $T(V)=\mathbb{C} \oplus V \oplus(V \otimes V) \oplus\left(V^{\otimes 3}\right) \oplus \ldots$ and the exterior algebra is the quotient

$$
\Lambda(V)=T(V) / J
$$

where $J$ is the ideal of $T(V)$ generated by $\{x \otimes x \mid x \in V\}$. We write the product in $\Lambda(V)$ as $(x, y) \mapsto x \wedge y$. The elements of $\Lambda(V)$ are called alternating tensors:
Lemma 1.1. For all $x, y \in V$ we have $x \wedge y=-y \wedge x$.

## Exercise 1. Proof Lemma 1.1

The k-th exterior power $\Lambda^{k}(V)$ is the subspace of $\Lambda(V)$ spanned by the products $v_{1} \wedge v_{2} \wedge \cdots \wedge v_{k}$, $v_{i} \in V \forall i$,

$$
\Lambda(V)=\bigoplus_{k=0}^{\infty} \Lambda^{k}(V)
$$

(this is a finite sum since $\operatorname{dim} V<\infty)$. Let $e_{1}, \ldots, e_{n}$ be the natural basis of $V,\left(e_{i}\right)_{j}=\delta_{i j}$. Then the $e_{i_{1}} \wedge \cdots \wedge e_{i_{k}}$ with $1 \leq i_{1}<\cdots<i_{k} \leq n$ form a basis of $\Lambda^{k}(V)\left(\right.$ note that $\operatorname{dim} \Lambda^{k}(V)=\binom{n}{k}$ ).
For $x \in \Lambda^{k}(V)$ we write $p_{i_{1}, \ldots, i_{k}}(x)$ for the coefficient of $x$ in terms of this basis: $x=\sum_{i_{1}<\cdots<i_{k}} p_{i_{1}, \ldots, i_{k}} e_{i_{1}} \wedge$ $\cdots \wedge e_{i_{k}}$. In particular, the $p_{i_{1}, \ldots, i_{k}}$ are linear maps $\Lambda^{k}(V) \rightarrow \mathbb{C}$.
An element $x \in \Lambda^{k}(V)$ is decomposable or pure if $x=v_{1} \wedge \cdots \wedge v_{k}$ where $\left\{v_{1}, \ldots, v_{k}\right\}$ is a linearly independent set of vectors in $V$.

Exercise 2. Check that the vectors $v_{1}, \ldots, v_{k}$ are linearly dependent if and only if $v_{1} \wedge \cdots \wedge v_{k}=0$.
Let $v_{1}, \ldots, v_{k}$ be linearly independent vectors, $k \leq n$. We can use them to form a $k \times n$ matrix $M=\left(M_{i j}\right)_{i j}$ of rank $k$ where $M_{i j}=\left(v_{i}\right)_{j}$, taking the $v_{i}$ as rows.
We use the following notation for the minor of $M$ of rows $a_{1}, \ldots, a_{r}$ and columns $b_{1}, \ldots, b_{r}$ (for $1 \leq r \leq k)$ :

$$
\triangle_{b_{1}, \ldots, b_{r}}^{a_{1}, \ldots, a_{r}}(M)
$$

Lemma 1.2. Let $x=v_{1} \wedge \cdots \wedge v_{k} \in \Lambda^{k}(V)$ be decomposable, $M$ as above. Then we have

$$
p_{i_{1}, \ldots, i_{k}}(x)=\triangle_{i_{1}, \ldots, i_{k}}^{1,2, \ldots, k}(M)
$$

Exercise 3. Prove Lemma 1.2. Hint: use the expansion of the $v_{i}$ in terms of $e_{1}, \ldots, e_{n}$.

### 1.2 The Grassmannian

Let $1<k<n$. The Grassmannian $\operatorname{Gr}(k, n)$ is the set of $k$-dimensional subspaces of $V=\mathbb{C}^{n}$. Take $U \in \operatorname{Gr}(k, n)$ and $\left\{v_{1}, \ldots, v_{k}\right\}$ a basis of $U$. Consider

$$
w:=v_{1} \wedge \cdots \wedge v_{k} \in \Lambda^{k}(V)
$$

(Since the $v_{i}$ 's are linearly independent, $w \neq 0$, a decomposable alternating tensor).
Note that $w$ does not depend on the choice of basis, up to multiplication by a non-zero scalar. If we associate to $w$ all the coefficients $p_{i_{1}, \ldots, i_{k}}(w)$, we get a well-defined element $\left(p_{i_{1}, \ldots, i_{k}}(w)\right)_{i_{1}<\cdots<i_{k}}$ of the projective space $\mathbb{P}^{N}$ for $N=\binom{n}{k}-1$. This gives us a map

$$
\varphi: \operatorname{Gr}(k, n) \rightarrow \mathbb{P}^{N}
$$

The $p_{i_{1}, \ldots, i_{k}}$ are called the Plücker coordinates. Note that in the definition of $\varphi$ we have chosen the indices to be strictly increasing.
We want to describe the image of $\varphi$. For this, we extend the definition of the $p_{i_{1}, \ldots, i_{k}}$ to arbitrary (multi) sets $\left\{i_{1}, \ldots, i_{k}\right\}$ with $i_{j} \in[n]=\{1, \ldots, n\}$ by setting $p_{i_{1}, \ldots, i_{k}}=0$ if there are $r \neq s$ such that $i_{r}=i_{s}$ and by setting $p_{i_{1}, \ldots, i_{k}}=\operatorname{sgn}(\pi) p_{j_{1}, \ldots, j_{k}}$ in case the $i_{1}, \ldots, i_{k}$ are distinct, $\left\{i_{1}, \ldots, i_{k}\right\}=\left\{j_{1}, \ldots, j_{k}\right\}$ with $1 \leq j_{1}<\cdots<j_{k} \leq n$ and $\pi$ is the permutation with $\pi\left(i_{k}\right)=j_{k}$ for all $k$. With this notation, we can describe the relations, the image of $\operatorname{Gr}(k, n)$ under $\varphi$ will satisfy.

The Plücker relations for $\operatorname{Gr}(k, n)$ are the relations

$$
\begin{equation*}
\sum_{r=0}^{k}(-1)^{r} p_{i_{1}, \ldots, i_{k-1}, j_{r}} p_{j_{0}, \ldots, \hat{j}_{r}, \ldots, j_{k}} \tag{1}
\end{equation*}
$$

where the sum is taken over all tuples $\left(i_{1}, \ldots, i_{k-1}\right),\left(j_{0}, \ldots, j_{k}\right)$ satisfying $1 \leq i_{1}<\cdots<i_{k-1} \leq n$ and $1 \leq j_{0}<\cdots<j_{k} \leq n$.

Exercise 4. Write the Plücker relations for $\operatorname{Gr}(2,5)$.
Facts. 1. $x \in \Lambda^{k}(V)$ is decomposable $\Longleftrightarrow$ the Plücker relations on $x$ are 0 ;
2. The image $\operatorname{im} \varphi \subseteq \mathbb{P}^{N}$ are the elements of $\mathbb{P}^{N}$ for which the Plücker relations are 0 ;
3. $\varphi: \operatorname{Gr}(k, n) \rightarrow \mathbb{P}^{N}$ is injective. It is called the Plücker embedding;
4. $\operatorname{im} \varphi$ is an irreducible projective variety, so $\operatorname{Gr}(k, n)$ is an irreducible projective variety.

For the proofs: (2) follows from (1) and from the definition of $\varphi$. For (1)-(3): [Jac96, §3.4] for (2),(3) [MS05, §14], for (4) W. Fulton, 1997, [Ful97, §8, §9.]

Example 1.3. For $\operatorname{Gr}(2,4)$, there is a single Plücker relation:

$$
p_{12} p_{34}-p_{13} p_{24}+p_{14} p_{23}
$$

Remark 1.4. Let $Y \subseteq \mathbb{P}^{r}$ be a projective variety, $R:=\mathbb{C}\left[x_{0}, x_{1}, \ldots, x_{r}\right]$ a graded ring, each $x_{i}$ of degree 1. The homogenous ideal $J(Y)$ of $Y$ is the ideal of $R$ generated by the homogenous elements of $R$ which vanish on $Y$. The homogenous coordinate ring of $Y$ is $\mathbb{C}[Y]:=R / J(Y)$. There is a natural projection

$$
\operatorname{pr}: \mathbb{C}^{r+1} \backslash\{0\} \rightarrow \mathbb{P}^{r}, \quad\left(a_{0}, \ldots, a_{r}\right) \mapsto\left[a_{0}: \ldots: a_{r}\right]
$$

The affine cone $C(Y)$ over $Y$ is the preimage of $Y$ under pr:

$$
C(Y):=\operatorname{pr}^{-1}(Y) \cup\{0\}
$$

One can show that $C(Y)$ is an affine variety whose ideal is $J(Y)$ regarded as an ideal of $R$ without the grading; the coordinate ring of $C(Y)$ coincides with the homogenous coordinate ring of $Y$, Har77, Exercise 2.10]. From the facts above, we get that the affine cone of $\operatorname{Gr}(k, n)$ can be identified with the decomposable elements of $\Lambda^{k}(V)$ together with 0 . And the coordinate ring of the affine cone of $\mathbb{C}[\operatorname{Gr}(k, n)]$ is the quotient of the polynomial ring in generators $x_{i_{1}, \ldots, i_{k}}$ with $1 \leq i_{1}<\cdots \leq i_{k} \leq n$ by the ideal generated by the Plücker relations. For details: Har77, §2]

### 1.3 Cluster algebra structure for $\operatorname{Gr}(2, n)$

For $k=2$, the Plücker relations are

$$
p_{i, j_{0}} p_{j_{1}, j_{2}}-p_{i, j_{1}} p_{j_{0}, j_{2}}+p_{i, j_{2}} p_{j_{0} \cdot j_{1}}
$$

where $1 \leq i \leq n, 1 \leq j_{0}<j_{1}<j_{2} \leq n$. We can rewrite these as

$$
\begin{equation*}
p_{a b} p_{c d}-p_{a c} p_{b d}+p_{a d} p_{b c} \quad \text { for all } a, b, c, d \text { with } 1 \leq a<b<c<d \leq n \tag{2}
\end{equation*}
$$

Exercise 5. Check the above.

Using this and the facts from above, we get the following result:
Lemma 1.5. The homogenous coordinate ring of $\operatorname{Gr}(2, n)$ is the quotient of the polynomial ring in variables $p_{a b}, 1 \leq a<b \leq n$, subject to the relations

$$
p_{a b} p_{c d}-p_{a c} p_{b d}+p_{a d} p_{b c} \quad \text { for all } 1 \leq a<b<c<d \leq n
$$

Remark 1.6. In this case, the Plücker coordinates can be parametrized by the diagonal and edges of a regular polygon $P_{n}$ with vertices $1,2, \ldots, n$, say clockwise: the coordinate $p_{a b}, a<b$, corresponds to the diagonal or boundary edge connecting $a$ and $b$. We can then interpret the relations (2) as "Ptolemy relations", e.g. $p_{14} p_{26}=p_{12} p_{46}+p_{16} p_{24}$ in the example $(n=6)$ :


### 1.4 The quiver of a triangulation of a surface

Let $S$ be a connected, oriented surface with boundary. Let $M \neq \emptyset$ be a finite set of marked points in $\bar{S}$. The points of $M$ are on the boundary or in the interior of $S$. Assume that $M$ contains at least one marked point on each boundary component and that ( $S, M$ ) is not one of the following:

$$
\left\{\begin{array}{l}
\text { sphere with } 1,2 \text { or } 3 \text { interior points } \\
\text { monogon with } 0 \text { or } 1 \text { interior points } \\
\text { digon or triangle with no interior points }
\end{array}\right.
$$

For details, we refer to [FST08].
We now restrict to the case where $(S, M)$ has no punctures. Consider simple non-contractible arcs in ( $S, M$ ), with endpoints in $M$ (up to isotopy fixing endpoints). An ideal triangulation of $(S, M)$ is a maximal collection of (isotopy classes of) such arcs which pairwise do not cross. Let $T$ be an ideal triangulation of $(S, M)$. Then we associate to $T$ a quiver $Q_{T}$ as follows.
The vertices of $Q_{T}$ are the arcs of $T$ and the boundary segments. We draw an arrow $i \rightarrow j$ if $i$ and $j$ are arcs of a common triangle of $T$ and $j$ is clockwise from $i$ (around a common endpoint) and if $i$ and $j$ are not both boundary segments. In the example, the quiver of a triangulation of a hexagon is drawn.
(1) In a triangulation, there can be loops (arcs starting and ending at the same marked
point). For an example, take an annulus with two points $P_{1}, P_{2}$ on the outer boundary and two on the inner boundary. You should be able to find a triangulation with an arc with both endpoints at $P_{1}$.
(2) Two different triangulations of the same surface ( $S, M$ ) can have the same quiver. Look at an annulus with one point on each boundary and try to find two different triangulations with the same quiver.

If $T$ is a triangulation of a polygon $P_{n}$, like below, there are $n$ boundary edges $(i, i+1)$. By definition of $Q_{T}$, there are no arrows between these. We will later consider quiver with arrows between vertices on the boundary.


In these lectures, the surface is always a regular polygon $P_{n}$ and $M=\{1,2, \ldots, n\}$ the set of vertices of $P_{n}$. So for $Q_{T}$, the vertices are just the diagonals of the triangulation and the boundary edges of the polygon (sometimes called "frozen vertices").

Theorem 1.7. [FZ03, Proposition 12.6] Let $n \geq 5$. Let $P_{n}$ be a convex $n$-gon. The homogenous coordinate ring $\mathbb{C}[\operatorname{Gr}(2, n)]$ of 2-planes in $n$-space is a cluster algebra:
$\left\langle p_{a b} \mid 1 \leq a<b \leq n\right\rangle /\{$ Ptolemy relations $\} \otimes \mathbb{C}=\mathbb{C}[\operatorname{Gr}(2, n)]$.
The cluster variables are the Plücker coordinates $p_{a b}$, where the $(a, b)$ are the diagonals in $P_{n}$ and the coefficients are the Plücker coordinates $p_{12}, p_{23}, \ldots, p_{n-1, n}, p_{1 n}$ (corresponding to the boundary edges of $P_{n}$ ).

The seeds are in bijection with the triangulations of $P_{n}$. The quiver of the seed is $Q_{T}$. Cluster mutation corresponds to the quadrilateral flip in a triangulation (and to the Ptolemy relations (2)

By the above result, $\mathbb{C}[\operatorname{Gr}(2, n)]$ can be regarded as a cluster algebra of type $\mathrm{A}_{n-3}$ with coefficients (cf. [Mar13, Ex 8.2.3]). It is sometimes called Ptolemy cluster algebra.

Exercise 6. Find $Q_{T}$ for the triangulation $T$ given by the diagonals (13), (35), (36), (16), (17) of an octagon.


### 1.5 The cluster algebra of a triangulated surface

Fomin and Zelevinsky introduced cluster algebras around 2002 as a framework for phenomena observed in their work on bases for enveloping algebras and on total positivity for algebraic groups. See [Zel07] for a short introduction.

Due to time constraints, we give a restricted definition of a cluster algebra here.
We take a triangulation $T$ of $(S, M)$. Let $T$ have $n$ arcs. We associate a variable $x_{i}, 1 \leq i \leq n$, for every diagonal of $T$. We associate variables $x_{n+1}, \ldots, x_{n+m}$ for every boundary segment. The tuple $\underline{x}=\left(x_{1}, \ldots, x_{n}, x_{n+1}, \ldots, x_{n+m}\right)$ is called a cluster and the pair $(\underline{x}, T)$ an (initial) seed. The cluster algebra of this seed is defined as a subring of $\mathcal{F} \mathbb{C}\left(x_{1}, \ldots, x_{n}, x_{n+1}, \ldots, x_{n+m}\right)$. The idea is to produce new variables through taking quotients of polynomials in these variables. The rule to do so can be given by the triangulation or more generally, by a quiver.

Consider an arc $i$ in $T$. Then $i$ belongs to two triangles in the triangulation and therefore determines a quadrilateral (4-gon) in the surface, for which $i$ is a diagonal. The flip of $i$ in the triangulation is the replacement by the arc $i$ by the other diagonal $i^{\prime}$ in this quadrilateral.

Definition 1.8. Let $T$ be a triangulation of $(S, M)$. We write $a, b, c, d$ for the variables of the four arcs or boundary segments of the quadrilateral given by $i$, writing clockwise around the boundary. Note that $a, b, c, d$ are elements of $\left\{x_{1}, \ldots, x_{n}, x_{n+1}, \ldots, x_{n+m}\right\} \backslash\left\{x_{i}\right\}$.


The mutation at $i$ of $\left(x_{1}, \ldots, x_{n}, \ldots, x_{n+m}\right)$ is the $n$-tuple $\left(x_{1}, \ldots, x_{i-1}, x_{i}^{\prime}, x_{i+1}, \ldots, x_{n}, \ldots, x_{n+m}\right)$ where $x_{i}^{\prime}$ is given by the relation

$$
x_{i} x_{i}^{\prime}=a c+b d
$$

The new variable $x_{i}^{\prime}$ is an element of $\mathcal{F}$. The mutation of the triangulation $T$ at the arc $i$ is the triangulation obtained by replacing the arc $i$ with the arc $i^{\prime}$ corresponding to the other diagonal in the quadrilateral defined by $i$.
The mutation of $T$ at $i$ is the new triangulation, i.e. $\mu_{i}(T)=T^{\prime}$ has arcs $1,2, \ldots, i-1, i^{\prime}, i+1, \ldots, n$.
Note that by construction, $x_{i}^{\prime}$ is an element of $\mathcal{F}$. Mutation can be done at any of the variables corresponding to the arc of the triangulation. And it can be iterated. note that mutation is involutive: if we mutate at $i$ and then directly afterwards at $i^{\prime}$, we go back to the original variables and triangulation.

Definition 1.9. Let $\mathcal{X} \subset \mathcal{F}$ be the set of all variables obtained from $\left\{x_{1}, \ldots, x_{n}, \ldots, x_{n+m}\right\}$ by arbitrary sequences of mutations. The cluster algebra of the triangulation $T$ of $(S, M)$ is defined to be the subalgebra $\mathcal{A}$ of $\mathcal{F}$ generated by $\mathcal{X}$,

$$
\mathcal{A}((S, M), T)=\langle\mathcal{X}\rangle_{\mathbb{C}} \subset \mathcal{F}
$$

## Example 1.10.



The cluster algebra given by a triangulation of a pentagon can be described explicitly. Let $x_{1}, x_{2}$ be the variable of the triangulation on the top in the figure below. For simplicity, we set the boundary
variables $y_{1}, \ldots, y_{5}$ all to be equal to 1 . In the figure, these are indicated in blue. We then work in $\mathbb{C}\left(x_{1}, x_{2}\right)$.
If we mutate first at the arc 2 , we consider the quadrilateral with the boundary vertices $1,2,3,4$. The mutation rule tells us that

$$
x_{2}^{\prime}=\frac{1+x_{1}}{x_{2}}
$$

In the next step, we mutate the arc 1 . Here, we get

$$
x_{1}^{\prime}=\frac{1+x_{2}^{\prime}}{x_{1}}=\frac{1+\frac{1+x_{1}}{x_{2}}}{x_{1}}=\frac{1+x_{1}+x_{2}}{x_{1} x_{2}}
$$

Then we mutate the arc with variable $x_{2}^{\prime}$ and get

$$
x_{2}^{\prime \prime}=\frac{1+x_{1}^{\prime}}{x_{2}^{\prime}}=\frac{1+\frac{1+x_{1}+x_{2}}{x_{1} x_{2}}}{\frac{1+x_{1}}{x_{2}}}=\left(\frac{1+x_{1}+x_{2}+x_{1} x_{2}}{x_{1} x_{2}}\right) \frac{x_{2}}{1+x_{1}}=\frac{1+x_{2}}{x_{1}}
$$

One computes that $x_{1}^{\prime \prime}=x_{2}$ and that if one mutates $x_{2}^{\prime \prime}$ next, one obtaines $x_{1}$. you should check this!.

There are a few things to observe here: In the definition of a cluster algebra, iterated fractions appear and so one could expect to get complicated rational functions. However, in the example, the denominators of all the variables obtained are a monomial in one or both of the initial variables. Also, the coefficients are integers. And only finitely many variables are produced through the mutations. These observations are due to the three main theorems on cluster algebras (stated in restricted form for simplicity - without coefficients and for surfaces.).

Theorem 1.11 (Laurent phenomenon). If we set $x_{n+1}=\ldots x_{n+m}=1$, then every cluster variable is a Laurent polynomial in $\mathbb{Z}\left[x_{1}^{ \pm}, \ldots, x_{n}^{ \pm}\right]$.

Theorem 1.12 (Finite type classification). $\mathcal{A}$ has finitely many variables if and only if it arises from a triangulation of a polygon.

Theorem 1.13 (Positivity). If we set $x_{n+1}=\ldots x_{n+m}=1$, then the cluster variables are all in $\mathbb{Z}_{>0}\left[x_{1}^{ \pm}, \ldots, x_{n}^{ \pm}\right]$.

The first two results are due to Fomin and Zelevinsky, in their first papers on cluster variables. The third had been a conjecture for over a decade and is now proved for general cluster variables by work of Lee and Schiffler and of Gross, Hacking, Keel and Kontsevich.

Remark 1.14. A more general definition of a cluster algebra is via quivers. For this, let $Q$ be a finite quiver without 2-cycles and without loops. Let $Q_{0}=\{1,2, \ldots, n, n+1, \ldots, n+m\}$ be the vertices of $Q$, for some $n>0$ and $m \geq 0$. The vertices $1, \ldots, n$ are mutable, the vertices $n+1, \ldots, n+m$ are frozen. We consider $\mathcal{F}=\mathbb{C}\left(x_{1}, \ldots, x_{n}, x_{n+1}, \ldots, x_{n+m}\right)$ where the tuple $\underline{x}=\left(x_{1}, \ldots, x_{n}, x_{n+1}, \ldots, x_{n+m}\right)$ of variables is called a cluster. The pair $\left(\left(x_{1}, \ldots, x_{n}, x_{n+1}, \ldots, x_{n+m}\right), Q\right)$ is a seed.

For $1 \leq i \leq n$, we define the mutation $\mu_{i}$ of the cluster in direction $i$ as follows: $\mu_{i}\left(x_{j}\right)=x_{j}$ for all $j \neq i$ and $\mu_{i}\left(x_{i}\right)=x_{i}^{\prime}$ where $x_{i} x_{i}^{\prime}=\prod_{j \rightarrow i} x_{j}+\prod_{j \leftarrow i} x_{j}$.
The mutation of $Q$ indirection $i, i \leq n$, is defined as follows:
(1) For any 2-path $j_{1} \rightarrow i \rightarrow j_{2}$ add an arrow $j_{1} \rightarrow j_{2}$ to $Q$ ("add shortcuts")
(2) invert all arrows incident with $i$
(3) remove 2-cycles.

Let $\mathcal{X}$ be the union of all variables obtained from $\underline{x}$ through arbitrary sequences of mutations. The cluster algebra of $(Q, \underline{x})$ is defined to be the subalgebra $\mathcal{A}$ of $\mathcal{F}$ generated by $\mathcal{X}$,

$$
\mathcal{A}(\underline{x}, Q)=\langle\mathcal{X}\rangle_{\mathbb{C}} \subset \mathcal{F}
$$



Figure 1: Untwisting and twisting moves in a Postnikov diagram

### 1.6 Cluster algebra structure for $\operatorname{Gr}(k, n)$

From now on we assume $k \leq \frac{n}{2}$.
In order to find a cluster algebra structure for arbitrary $k$, we will use the notion of a "diagram in a disk with $n$ marked points" instead of a triangulation of a polygon. We write $S_{n}$ for the set of permutations of $n$. We write $D_{n}$ for a disk with $n$ marked points $\{1,2, \ldots, n\}$ on the boundary (going clockwise). The following definition is due to Postnikov, Pos06].

Definition 1.15. Let $\sigma \in S_{n}$ be a permutation. An alternating strand diagram or Postnikov diagram of type $\sigma \in S_{n}$ in the disk $D_{n}$ with vertices $1, \ldots, n$, is a collection of $n$ oriented strands (smooth curves) $\gamma_{1}, \ldots, \gamma_{n}$ (up to isotopy) in $D_{n}$ with $\gamma_{i}$ starting and $i$ and ending at $\sigma(i)$ satisfying
(a) The $\gamma_{i}$ have no self-intersections;
(b) There are finitely many intersections and they are transversal, of multiplicity 2 ;
(c) Crossings alternate (following any strand, the strands crossing it alternate between crossing from the left and crossing from the right);
(d) There are no "unoriented lenses": if two strands cross, they form an oriented disk.

For an illustration we refer to Figure 2 Postnikov diagrams can be simplified under two types of reductions as in Figure [1] also called twisting and untwisting moves. The moves obtained by reflecting the diagrams of Figure 1 in a horizontal line are also allowed. A Postnikov diagram is called reduced if it cannot be simplified with untwisting moves.

Remark 1.16. The conditions can be relaxed, i.e. if the surface has interior marked points or if it has several boundary components (like an annulus). Conditions (a) and (d) will no longer hold then. Strand diagrams appear as "webs" in [Gon17]. Strand diagrams for orbifolds are introduced in [BPV].

Any Postnikov diagram divides the surface into alternating and oriented regions. We label the alternating regions by $i$ whenever $\gamma_{i}$ is on the right of the region.
Let $\sigma_{k, n} \in S_{n}$ be the permutation $i \mapsto i+k$ (reducing modulo $n$ ).
Proposition 1.17. [Pos06, Proposition 5]
(a) Any $\sigma_{k, n}$-diagram in $D_{n}$ has $k(n-k)+1$ alternating regions, $(k-1)(n-k-1)$ internal ones, $n$ on the boundary.
(b) Each label is a $k$-subset of $[n]$.
(c) Every $k$-subset of $[n]$ appears as a label in a $\sigma_{k, n}$-diagram on $D_{n}$.

Note that if $\sigma=\sigma_{k, n}$, the $n$ boundary alternating regions in (a) above have the labels $[1, k],[2, k+1]$, $\ldots,[n, n+k]$ (reducing modulo $n$ ), see Figure 2,

Example 1.18. Figure 2 shows an example of a Postnikov diagram of type $\sigma_{3,7}$ in $D_{7}$.


Figure 2: $\mathrm{A} \sigma_{3,7}$ Postnikov diagram with its labels


Figure 3: Orientation convention for the quiver $Q(D)$, on the right for the boundary.

### 1.7 The quiver of a Postnikov diagram

Let $D$ be a $\sigma_{k, n}$-diagram. Each label of $D$ gives a Plücker coordinate, so we can associated to $D$ a collection of Plücker coordinates, $D \mapsto \tilde{p}(D)=p(D) \cup C$ where $C$ are the Plücker coordinates of the boundary alternating regions, i.e. the $p_{i, i+1, \ldots, i+k-1}$ for $i=1, \ldots, n$. With this, we can associate a quiver $Q(D)$ to $D$.

This is similar to quiver of a triangulation (use remark on bijection 1.22). The boundary convention is however different as we will see.

Definition 1.19. Let $D$ be a Postnikov diagram Then the quiver $Q(D)$ of $D$ has as vertices the $k$-subsets of $D$. The frozen vertices are the $k$-subsets of the boundary alternating regions of $D$. The arrows of $Q(D)$ are given by the "flow": Whenever two $k$-subsets are separated by only two crossing strands, there is an arrow between them, following the orientation of the strands, see Figure 3 At the end, we remove all 2-cycles that may habe appeared through this.

Example 1.20. The quiver of the Postnikov diagram from Example 1.18 is in Figure 4

Note that the quiver of a Postnikov diagram contains arrows between boundary vertices, unlike the quiver of a triangulation.

Definition 1.21. To any Postnikov diagram $D$ of type $\sigma_{k, n}$, one can define a cluster algebra: The initial seed is given by the set $\left\{x_{I}\right\}_{I \in \tilde{p}(D)}$ and the quiver $Q(D)$. The cluster algebra $\mathcal{A}(D):=$ $\mathcal{A}\left(\left\{x_{I}\right\}_{I \in \tilde{p}(D)}, Q(D)\right)$ is the $\mathbb{C}$-subalgebra of $\mathbb{C}\left(\left(x_{I}\right)_{I}\right)$ generated by the $x_{I}$ with $I \in C$ and by the $x_{I}$, $I \in p(D)$ and all elements obtained from the latter under arbitrary sequences of mutations.

Each $\sigma_{k, n}$-diagram gives rise to a seed in $\mathcal{A}(D)$. Scott proves [Sco06, Theorem 3] that there is an isomorphism $\varphi: \mathcal{A}(D) \xrightarrow{\sim} \mathbb{C}[G r(k, n)]$ sending $x_{I}$ to $p_{I}$ for any $k$-subset $I \in \tilde{p}(D)$. In other words, $\mathbb{C}[G r(k, n)]$ can be viewed as a cluster algebra where each Plücker coordinate is a cluster variable and where the $\sigma_{k, n}$-diagrams give some of its seeds.


Figure 4: The quiver of the Postnikov diagram in Figure 2


Figure 5: Modified version of Scott's construction. The dotted lines indicate boundary edges

Note that here we do not invert coefficients (see [Mar13, §9]). If we invert the coefficients, the cluster algebra is the coordinate ring of the Zariski-open subset of the Grassmannian defined by non-vanishing of the coefficients $p_{1, \ldots, k}, p_{2, \ldots, k+1}, \ldots p_{n, 1, \ldots, k-1}$.

Remark 1.22 (Scott map). Let $k=2$ and $n \geq 4$. Then there is a bijection between

$$
\left.\{\text { Triangulations of a convex } n \text {-gon }\} \quad \stackrel{1: 1}{\longleftrightarrow} \text { \{reduced } \sigma_{2, n} \text {-diagams }\right\}
$$

arising from equipping each triangle in the triangulation with strand segments as shown in Figure 5 with a slight modification of Scott's definition from [Sco06 Section 3] in order for strands to end at the boundary vertices.
Exercise 7. Draw the $\sigma_{2,7}$-diagram $D$ for the triangulation $T$ from Exercise6. Compare the two quivers $Q_{T}$ and $Q(D)$.
Remark 1.23. A correspondence as the one given by Scott, see Remark 1.22 is not known for arbitrary $k$. The only other cases where a combinatorial construction is available are $\sigma_{k, n}$ diagrams for $n=2 m$ and $k=m+1$; these arise from rhombic tilings, [ $\operatorname{Cos}$ ].

By Remark 1.22, for $k=2$, the $\sigma_{2, n}$-diagrams correspond to triangulations of polygons and thus in this case, the Postnikov diagrams give all the seeds (Theorem 1.7). If $D$ is $\sigma_{3, n}$-diagram for $n \in\{6,7,8\}$ then the cluster algebra $\mathcal{A}(D)$ of any $\sigma_{3, n}$-diagram is of finite type, i.e. there are only finitely many cluster variables. However, not all seeds $\mathcal{A}(D)$ arise from $\sigma_{3, n}$-diagrams.

In all other cases (with $k \leq \frac{n}{2}$ ), the cluster algebra $\mathcal{A}(D)$ has infinitely many cluster variables.

## 2 Dimer models on surfaces and associated algebras

### 2.1 Dimer models with boundary

Dimer models with boundary have been introduced in [BKM16].

Definition 2.1. (1) A dimer model (with boundary) is a finite quiver $Q$ that embeds into a surface $S$ such that each connected component of $S \backslash Q$ is simply connected and bounded by an oriented cycle.
(2) The cycles bounding the connected components of $S \backslash Q$ are called the unit cycles. The arrows of $Q$ are internal if they are contained in two faces and boundary if they are contained in one face. The vertices incident with boundary arrows are called boundary vertices.

In the case without boundary, the dimer model have been studied by various authors, e.g. [Dav11], [Boc12], [Bro12]. The boundary convention has also been used independently in [BIRS11], [Fra12] and DL16a.

Example 2.2. The following quiver is an example of a dimer model with boundary, on a disk with 6 boundary vertices.


Example 2.3. (a) The quiver of any Postnikov diagram is a dimer model in the disk.
(b) Let $T$ be a triangulation of a polygon $P_{n}$. If we complete the quiver $Q_{T}$ by $n-2$ clockwise and 2 anti clockwise arrows between the boundary vertices in order to close all the cycles involving boundary vertices, we get a dimer model in $P_{n}$. We denote the resulting quiver by $Q(T)$.


Exercise 8. Find a triangulation $T$ such that $Q(T)$ as defined in Example 2.3 (b) is the dimer of Example 2.2.

### 2.2 Algebras associated to dimer models

Most of the background for this section can be found in [BKM16] and [JKS16]. We associate associate algebras to dimer models. For this, we first describe the relations on paths we will use. If $\alpha$ is an internal arrow of a dimer model with boundary, then there are exactly two paths $\alpha^{ \pm}$back from its head to its tail. For example, in the dimer model from Example 2.3(b), for the arrow $\alpha: 14 \rightarrow 15$, the two paths back are $p_{\alpha}^{+}:(15 \rightarrow 45) \circ(45 \rightarrow 14)$ and $p_{\alpha}^{-}$is $(15 \rightarrow 16) \circ(16 \rightarrow 12) \circ(12 \rightarrow 13) \circ(13 \rightarrow 14)$,
composing arrows left to right:


Definition 2.4. Let $Q$ be a dimer model with boundary. The dimer algebra $A_{Q}$ of $Q$ is quotient of the path algebra of $\mathbb{C} Q$ by the relations $p_{\alpha}^{+}=p_{\alpha}^{-}$for every internal arrow of $Q$.

We may also take the completed version of this path algebra, see Remark 2.11 below.
Another way to describe the relations for the definition of the dimer algebra is as follows. Let $W$ be the (natural) potential associated to $Q$ :

$$
W:=W(Q):=\sum_{p \text { pos. unit cycle }} p-\sum_{p \text { neg. unit cycle }} p .
$$

Then the relations $p_{\alpha}^{+}=p_{\alpha}^{-}$arise from taking all cyclic derivatives $\partial(W)$ of $W$ with respect to internal arrows.

Observe that $A_{Q}$ is an infinite dimensional algebra.
Exercise 9. Any two unit cycles at a vertex of a dimer model $Q$ commute. Why is this?
Remark 2.5. Let $i \in Q_{0}$ be a vertex of a dimer model. Let $U_{i}$ be a unit cycle at $i$. Then $t:=\sum_{i \in Q_{0}} U_{i}$ is a central element of $A_{Q}$ since any two unit cycles at $i$ commute. So we get $\mathbb{C}[t] \subseteq Z\left(A_{Q}\right)$.

Let $e_{1}, e_{2}, \ldots, e_{n}$ be the idempotent elements of $A_{Q}$ corresponding to the boundary vertices of $Q$ and let $e:=e_{1}+\cdots+e_{n}$.

Definition 2.6. Let $Q$ be a dimer model with boundary. The boundary algebra of $Q$ is the idempotent subalgebra $B_{Q}:=e A_{Q} e$.
$B_{Q}$ has as basis all paths of $Q$ between boundary vertices (up to the relations). Let $t:=U_{1}+\cdots+U_{n}$ be the sum of the unit cycles at the boundary vertices. Then $\mathbb{C}[t] \subseteq Z\left(B_{Q}\right)$.

Note that there is also a completed version of this, see Remark 2.11
Example 2.7. The boundary algebra from Example 2.2 is given by the following quiver:



Figure 6: The quiver $\Gamma_{n}$

One can show that

$$
B_{Q}=\mathbb{C}\left[x_{i}, y_{i} \mid i=1, \ldots, n\right] /\left\langle\left\{x y-y x, x^{2}-y^{4}\right\}\right\rangle
$$

and that $Z\left(B_{Q}\right)=\mathbb{C}[t]$.
Note that $\left\{x y-y x, x^{2}-y^{4}\right\}$ is short for the twelve relations $x_{i} y_{i}-y_{i-1} x_{i-1}, x_{i} x_{i+1}-y_{i-1} y_{i-2} y_{i-3} y_{i-4}$ for $i=1, \ldots, 6$ and where the indices are reduced modulo 6 .

The boundary algebra $B_{Q}$ is also an infinite dimensional algebra, in general its global dimension is infinite. $A_{Q}$ and $B_{Q}$ are not well understood apart from the case where the surface $S$ is a disk and the dimer models arise from Postnikov diagrams of type $\sigma_{k, n}$ or partly from the cases where $S$ is a surface and $Q$ arise from a triangulation of $S$.
From now on we restrict to dimer models on disks. As we are interested in the cluster structure of the Grassmannian, we will also restrict the dimer models: we want them to correspond to $\sigma_{k, n}$-diagrams. Recall that any Postnikov diagram $D$ on a disk determines a quiver $Q(D)$ and that this is a dimer model with boundary. There is also a way to go from dimer models to Postnikov diagrams ([Pos06, §14]): For any arrow in the dimer model, draw two segments of oriented curves, crossing on the arrow, pointing in the same direction as the arrow, then connect these; the strands correspond to zig-zag paths in the disk, cf. [Bro12 Section 5].

Exercise 10. Find the Postnikov diagram for the dimer model above. Determine its permutation.
Definition 2.8. If $Q$ is a dimer model on a disk such that the associated Postnikov diagram is of type $\sigma_{k, n}$, we call $Q$ a $(k, n)$-dimer.

Remark 2.9. - The $(2, n)$-dimers correspond to $Q_{T}, T$ a triangulation of an $n$-gon.

- $(k, n)$-dimer exist for any $(k, n)$. Examples of such arise from the rectangular $\sigma_{k, n}$-diagrams of Scott, [Sco06]
Theorem 2.10 ([BKM16]). Let $Q$ and $Q^{\prime}$ be two $(k, n)$-dimers. Let $e=e_{1}+\cdots+e_{n}$ the sum of the boundary idempotents. Then

$$
e A_{Q} e \cong e A_{Q^{\prime}} e \cong \mathbb{C} \Gamma_{n} /\left\langle\left\{x y-y x, x^{k}-y^{n-k}\right\}\right\rangle
$$

where $\Gamma_{n}$ is the quiver in Figure 6 and the relations are with indices as in Example 2.7 .
Let $B:=B_{k, n}:=\mathbb{C} \Gamma_{n} /\left\langle\left\{x y-y x, x^{k}-y^{n-k}\right\}\right\rangle$. Then the boundary algebra of any $(k, n)$-dimer model is isomorphic to $B$.

Remark 2.11. The above theorem is about the completed versions of these algebras. For $A_{Q}$, the algebra $\widehat{A}_{Q}$ is the completion with respect to $(U)$ for $U=\sum_{i \in Q_{0}} U_{I}$ and $\widehat{B}$ is the completion of $B$ with respect to $(t), t=\sum_{i=1}^{n} x_{i} y_{i}$. Note that the completion $\widehat{B}$ of $B$ with respect to $(t)$ is the same as the completion with respect to the arrow ideal $m$, since $\left(m_{A_{Q}}\right)^{N_{1}} \subseteq(U) \subseteq m_{A}$ and $\left(m_{B}\right)^{N_{2}} \subseteq(t) \subseteq m_{B}$ for some $N_{1}, N_{2}$, [JKS16, §3]. In particular, we have $e \widehat{A_{Q}} e \cong \widehat{B}$. See [BKM16, §11].
The centres of $B$ and $\widehat{B}$ are polynomial rings, $Z(B)=\mathbb{C}[t]$ and $Z(\widehat{B})=\mathbb{C}[|t|]$.

We will from now on work with the completed versions as we want the Krull-Schmidt Theorem to hold in the module categories. To simplify notation, we will write $B$ and $A_{Q}$ instead of $\widehat{B}$ and $\widehat{A_{Q}}$ to simplify notation.
We consider $B$-modules which are free over the centre and define

$$
\begin{aligned}
\mathcal{F}_{k, n}:=\mathrm{CM}(B) & :=\{M B \text {-module } \mid M \text { is free over } Z(B)\} \\
& =\left\{M \mid \operatorname{Ext}_{B}^{i}(M, B)=0 \text { for all } i>0\right\}
\end{aligned}
$$

Thee categories $\mathcal{F}_{k, n}$ have been introduced by Jensen-King-Su. In JKS16], the authors prove that $\mathcal{F}_{k, n}$ provides an additive categorification of Scott's cluster algebra structure on the Grassmannian.

## 3 Grassmannian cluster categories

As before, let $k \leq \frac{n}{2}$. It is our goal to understand the categories $\mathcal{F}_{k, n}$ better, for example, to give a description of (some of) their indecomposable modules. The rank of a module $M \in \mathcal{F}_{k, n}$ is defined to be $\frac{1}{n} r \mathrm{k}_{Z} M$. Among the indecomposable modules, the rank 1 modules play a special role as they appear in filtrations of higher rank modules. Also, the rank 1 modules are well understood: As is shown in [JKS16, §5], there is a bijection between indecomposable rank 1 modules in $\mathcal{F}_{k, n}$ and $k$-subsets of $\{1,2, \ldots, n\}$. As we know, these are in bijection with Plücker coordinates and hence with certain cluster variables, see Section 1.6 . We write $\mathbb{M}_{I}$ for the indecomposable rank 1 module with $k$-subset $I$ of $[n]$.

### 3.1 Rank 1 modules

We first describe how $B$ acts on the rank 1 modules and then show how these modules can be understood as a lattice diagram with vertices for basis elements. Note that all modules in $\mathcal{F}_{k, n}$ are infinitedimensional as they are free over the centre.

Any rank 1 module is given by $n$ copies of the centre, $U_{1}, \ldots, U_{n}$, with $U_{i}:=\mathbb{C}[|t|]$. Consider $\mathbb{M}_{I}, I$ a $k$-subset of $[n]$. The actions of $x_{i}$ and of $y_{i}$ on the $U_{i}$ are as follows:

$$
x_{i}: U_{i-1} \rightarrow U_{i} \text { acts as }\left\{\begin{array} { c c } 
{ 1 } & { \text { if } i \in I } \\
{ t } & { \text { if } i \notin I }
\end{array} \quad y _ { i } : U _ { i } \rightarrow U _ { i - 1 } \text { acts as } \left\{\begin{array}{cc}
t & \text { if } i \in I \\
1 & \text { if } i \notin I
\end{array}\right.\right.
$$

As an example, we consider for $k=3$ and $n=7, I=\{2,4,5\}$. One way to illustrate $\mathbb{M}_{I}$ is by putting a vector space (the centre $Z$ ) at each vertex, writing $U_{i}$, and indicating the action of the arrow. We break the cycle up, to draw the vertices on a line:


Another way is to view this as a lattice diagram on a cylinder. For example, for $k=3$ and $n=7$, $I=\{2,4,5\}$. The arrows $x_{2}, x_{4}, x_{5}$ and the arrows $y_{1}, y_{3}, y_{6}, y_{7}$ all act as multiplication by 1 , the other arrows as multiplication by $t$.


The rim of a rank 1 module $\mathbb{M}_{I}$ is formed by the top vertices in its lattice diagram and by the arrows connecting them. If the rim of $\mathbb{M}_{I}$ has two successive arrows $y_{i}, x_{i+1}$ for some $i$, we say that the rim or $\mathbb{M}_{I}$ has a peak (at $i$ ).

Remark 3.1. In $\mathcal{F}_{2, n}$ every indecomposable object is a rank 1 module, i.e. of the form $\mathbb{M}_{i, j}$ for some $1 \leq i \neq j \leq n$. The objects $\mathbb{M}_{i, i+1}$ are the indecomposable projective-injectives objects.
Exercise 11. Show that $\operatorname{Hom}_{M}\left(\mathbb{M}_{I}, \mathbb{M}_{J}\right) \cong \mathbb{C}[|t|]$ for all $I, J$.
Definition 3.2. Let $I$ and $J$ be $k$-subsets of $[n]$. We say that $I$ and $J$ cross if the complete graph $K_{I \backslash J}$ on $I \backslash J$ intersects $K_{J \backslash I}$.

Note that $I$ and $J$ do not cross if and only if the two $k$-subsets appear together in a $\sigma_{k, n}$-diagram. One can prove that the maximal non-crossing collections are exactly the ones arising from $\sigma_{k, n}$-diagrams: By [Sco06, Corollary 1], the $k$-subsets of any $\sigma_{k, n}$-diagram form a maximal non-crossing collection and by [OPS15, Thm. 7.1], every such collection arises in this way.

For an example of two crossing 4 -subsets of [8] consider $I=\{1,4,6,8\}$ and $J=\{2,5,6,7\}$ :


3

Crossing subsets are exactly the ones giving rise to non-trivial extensions:
Proposition 3.3. [JKS16, Proposition 5.6]. Ext ${ }_{B}^{1}\left(\mathbb{M}_{I}, \mathbb{M}_{J}\right)=0$ of and only if $I$ and $J$ do not cross.
One can use this to find has cluster-tilting objects in $\mathcal{F}_{k, n}$ : let $Q$ be a $(k, n)$-dimer, let $T:=\oplus_{I \in Q_{0}} \mathbb{M}_{I}$. Then $T$ is a maximal rigid object in $\mathcal{F}_{k, n}$ ([JKS16, BKM16]).

### 3.2 Dimer models as combinatorial approach to cluster categories

Theorem 2.10 above is a consequence of the following result:
Theorem 3.4. [BKM16, Theorem 10.3] Let $Q$ be a ( $k, n$ )-dimer and $B$ as above, $e=e_{1}+\cdots+e_{n}$ the sum of the idempotents for the boundary vertices of $Q$. Then

$$
A_{Q} \cong \operatorname{End}_{B}(T)
$$

and hence $e A_{Q} e \cong B^{o p}$.

$$
\begin{aligned}
A_{Q} & \xrightarrow{g} \operatorname{End}_{B}(T)=\operatorname{Hom}_{B}\left(\oplus_{I \in Q_{0}} \mathbb{M}_{I}, \oplus_{I \in Q_{0}} \mathbb{M}_{I}\right) \\
e & \mapsto \mathrm{id}_{\mathbb{M}_{I}} \\
\alpha: I \rightarrow J & \mapsto \varphi_{I J}: \mathbb{M}_{I} \rightarrow \mathbb{M}_{J}(\text { "minimal codimension map" })
\end{aligned}
$$

and extend to $T$ (by 0 's).
(a) One shows that $g$ is an algebra homomorphism.
(b) For the surjectivity of $g$ : to find $g^{-1}\left(\varphi_{I J}\right)$ we need a "minimal" path $p_{I J}: I \rightarrow \cdots \rightarrow J$ in $Q$ (unique in $A_{Q}$ ). It is difficult to see that this maps to $\varphi_{I J}$. F0r this, we use weights on the arrows of $Q$, these are subsets of $[n]$, and show that the minimal path avoids at least one label of $[n]$ (using the proper ordering of strands around vertices).
(c) For the injectivity of $g$, we prove a Lemma stating that if $p: I \rightarrow J$ is a path in $Q$, then there exists $r \geq 0$ such that $p=u^{r} \circ p_{I J}$ where $u$ is any unit cycle at $J$. This yields injectivity: take $m \in A_{Q}$ with $g(m)=0$. Without loss of generality, $m=e_{I} A_{Q} e_{J}$. By the lemma, $m=\sum_{r=0}^{l} \lambda_{r} u^{r} p_{I J}$ for some coefficients $\lambda_{r}$ and so $g(m)=\sum_{r=0}^{l} \lambda_{r} u^{r} \varphi_{I J}=0$. Since the $t^{r} \varphi_{I J}$ are linearly independent elements of $\operatorname{Hom}_{B}\left(\mathbb{M}_{I}, \mathbb{M}_{J}\right)$, this implies $\lambda_{r}=0$ for all $r$.
(d) For the statement $e A_{Q} e \cong B^{o p}$, use the isomorphism in the theorem: $e A_{Q} e \cong g(e) \operatorname{End}_{B} T g(e)=$ $\operatorname{End}_{B} P=B^{o p}$

### 3.3 Categories of finite type

The category $\mathcal{F}_{k, n}$ has of finitely many indecomposables if and only if $k=2$ or $k=3$ and $n \in\{6,7,8\}$. The Auslander-Reiten quiver of $\mathcal{F}_{k, n}$ has as vertices the isomorphism classes of indecomposable objects and it has an arrow for every irreducible map. The dotted lines in the Auslander-Reiten quiver are the Auslander-Reiten translate, sending an object to its neighbour on the left. The Auslander-Reiten quiver provides a good understanding of the category, since the $\mathcal{F}_{k, n}$ are Krull-Schmidt, every object can be uniquely written as a direct sum of indecomposables. In the case $k=2$, the category $\mathcal{F}_{2, n}$ is a cluster category of type $\mathrm{A}_{n-3}$ (with projective-injective objects). Its Auslander-Reiten quiver sits on a Moebius strip. Recall that for $\mathcal{F}_{2, n}$, all the indecomposables are of the form $\mathbb{M}_{i, j}$ (Remark 3.1).

Example 3.5. The Auslander-Reiten quiver of $\mathcal{F}_{2,6}$ has the following form:


There are $\binom{6}{2}$ indecomposable modules, indexed by the 2-subsets of 6 . The projective-injective indecomposables are the $\mathbb{M}_{i, i+1}$ (reducing modulo 6 ), drawn in a box in the quiver.

The Auslander-Reiten quivers of the categories $\mathcal{F}_{3, n}$ have been described in [JKS16, Fig 10,11,12]. We recall them here. Some of the indecomposable objects are rank 1 modules, but there are also rank 2 modules ( $n=6,7,8$ ) and rank 3 modules ( $n=8$ ).

Example 3.6. Auslander-Reiten quiver of $\mathcal{F}_{3,6}$. It is formed by 4 slices of shape $D_{4}$


Dynkin diagram of type $D_{4}$
and 6 additional vertices, corresponding to the projective-injective modules $\mathbb{M}_{i, i+1, i+2}$ (reducing indices modulo 6).


There are $\binom{6}{3}=20$ rank 1 indecomposables and two rank 2 indecomposables, the latter indicated by a vertex • (the left most and the right most are identified). The projective-injectives are drawn in boxes.

Example 3.7. The Auslander-Reiten quiver of $\mathcal{F}_{3,7}$ is is formed by 7 slices of shape $E_{6}$


$$
\text { Dynkin diagram of type } E_{6}
$$

and 7 additional vertices, corresponding to the projective-injective modules $\mathbb{M}_{i, i+1, i+2}$ (reducing modulo 7). This figure shows part of it. To complete it, one can continue along the dotted lines, using the fact that $\tau^{-2}\left(\mathbb{M}_{i, j, k}\right)=\mathbb{M}_{i+3, j+3, k+3}$.


There are $\binom{7}{3}=35$ rank 1 indecomposables and 14 rank 2 indecomposables, indicated by

Example 3.8. The Auslander-Reiten quiver of $\mathcal{F}_{3,8}$ is formed by 16 slices of shape $E_{8}$

and 8 additional vertices, corresponding to the projective-injective modules $\mathbb{M}_{i, i+1, i+2}$ (reducing modulo 8 ), drawn in boxes. Apart from the rank 1 modules, there are rank 2 modules drawn as $\bullet$ and rank 3 modules drawn as $\boldsymbol{\square}$. The rest of the shape of the Auslander-Reiten quiver can be obtained using the fact that for rank 1 modules, $\tau^{-2}\left(\mathbb{M}_{i, j, k}\right)=\mathbb{M}_{i+3, j+3, k+3}$.


There are $\binom{8}{3}=56$ rank 1 indecomposables, 56 rank 2 indecomposables and 24 rank 3 indecomposables.

### 3.4 Structure of $\mathcal{F}_{k, n}$ in infinite types

The aim of the remainder of these notes is to provide more information about the categories $\mathcal{F}_{k, n}$ which have infinitely many indecomposables. We will describe part of the Auslander-Reiten quiver in the general situation. The main tool in this section is a result determining Auslander-Reiten sequences.
Remark 3.9. (i) If $M \in \mathcal{F}_{k, n}$ is indecomposable rigid, then it has a filtration $M \cong \begin{gathered}\mathbb{M}_{I_{1}} \\ \vdots \\ \mathbb{M}_{I_{d}}\end{gathered}$ by rank 1 modules $\mathbb{M}_{I_{1}}, \ldots, \mathbb{M}_{I_{d}}$ and this filtration is unique. The rank of $M$ is $d$.
(ii) Note that the rank is additive on Auslander-Reiten sequences.

There are certain canonical Auslander-Reiten sequences which involve rank 1 modules. If $\mathbb{M}_{I}$ is an indecomposable whose rim has $s$ peaks (see Section 3.1), then $\Omega\left(\mathbb{M}_{I}\right)$ is a rank $(s-1)$ module ([BB17]), where $\Omega$ is the syzygy functor. In particular, if $I=\{i, j, j+1, \ldots, j+k-2\}$ with $i+1 \neq j$ and $i-1 \neq j+k-2$, the rim of $\mathbb{M}_{I}$ has two peaks and $\Omega\left(\mathbb{M}_{I}\right)=\mathbb{M}_{J}$ is also a rank 1 module, with $J=\{i+1, i+2, \ldots, i+k-1, j+k-1\}$, cf. [BB17] §2].
Theorem 3.10 ([BBGE19]). Let $3 \leq k \leq \frac{n}{2}$ and $I=\{i, j, j+1, \ldots, j+i-2\}$ and $J$ such that $\Omega\left(\mathbb{M}_{I}\right)=\mathbb{M}_{J}$. Then there exists an Auslander-Reiten sequence

$$
\mathbb{M}_{I} \hookrightarrow \frac{\mathbb{M}_{X}}{\mathbb{M}_{Y}} \rightarrow \mathbb{M}_{J}
$$

with rigid middle term $\frac{\mathbb{M}_{X}}{\mathbb{M}_{Y}}$ for $X=\{i+1, j, \ldots, j+k-3, j+k-1\}$ and $Y=I \cup J \backslash X$.
The middle term is indecomposable if and only if $j \neq i+2$ and if $j=i+2, \frac{\mathbb{M}_{X}}{\mathbb{M}_{Y}}=P_{i} \oplus$ $\mathbb{M}_{\{i, i+2, \ldots, i+k-1, i+k+1\}}$ for $P_{i}=\mathbb{M}_{\{i+1, i+2, \ldots, i+k\}}$.
Remark 3.11. - To proves the rigidness of the middle term, one shows that $\operatorname{dimExt}{ }^{1}\left(\mathbb{M}_{I}, \mathbb{M}_{J}\right)=1$ (using the description of extensions between rank 1 modules from [BB17]).

- In case $k=3$, the theorem above covers all Auslander-Reiten sequences where both end terms are rank 1 modules.

Theorem 3.12. For $(k, n) \in\{(3,9),(4,8)\}$ the category $\mathcal{F}_{k, n}$ is tame, $\mathcal{F}_{3,9}$ has tubes of rank $2,3,6$ and $\mathcal{F}_{4,8}$ has tubes of rank $2,4,4$.

Using Theorem 3.10 and computing syzygies for rank 2 modules, we are able to determine the tubes with low rank rigid modules ([BBGE19]).

In all other infinite types, $\mathcal{F}_{k, n}$ is wild and so a complete characterisation is not expected. However, we have the following, [BBGEL22]:

Theorem 3.13. If $\mathcal{F}_{k, n}$ is of infinite type, then it is tubular.
The ranks of the tubes are factors of $2 \frac{\mathrm{lCm}(k, n)}{k}$.
Furthermore, if $M=\mathbb{M}_{I} / \mathbb{M}_{J}$ is rigid indecomposable, then the rims of $I$ and $J$ form exactly 3 boxes.
We expect that the rigid indecomposable rank two modules are exactly the ones where the rims of $I$ and $J$ form 3 boxes: in joint work with E. Yildirim, we have recently found a way to construct non-trivial extensions for indecomposable rank two modules where the rims do not form 3 boxes (in that case, there are at least 4 boxes or some of the regions are not boxes but so-called "quasi-boxes" ).

### 3.5 Periodicity in $\mathcal{F}_{k, n}$

For any $k$-subset $I$ of $[n], I=\left\{i_{1}, i_{2}, \ldots, i_{k}\right\}$, we define $I+k$ to be the $k$-subset $\left\{i_{1}+k, i_{2}+k, \ldots, i_{k}+\right.$ $k\}$. Let $\tau$ be the Auslander-Reiten translate in $\mathcal{F}_{k, n}$.
Example 3.14. Let $\mathbb{M}_{I}$ be an indecomposable in $\mathcal{F}_{k, n}$. Then $\Omega^{2}\left(\mathbb{M}_{I}\right)=\mathbb{M}_{I+k}$.
Proposition 3.15. Every indecomposable in $\underline{\mathcal{F}_{k, n}}$ is $\tau$-periodic with period a factor of $\frac{2}{k} \operatorname{lcm}(k, n)$.
The proof of this proposition uses $\mathcal{F}_{k, n} \simeq \mathrm{CM}^{\mathbb{Z}_{n}}\left(R_{k, n}\right)$ (graded Morita equivalence) for the ring $R_{k, n}:=\mathbb{C}[x, y] /\left(x^{k}-y^{n-k}\right)$ where $x$ has degree 1 and $y$ has degree -1 , cf. DL16b. Theorem 3.22]. Alternatively, one can use [Kel13 Theorem 8.3] to find $\tau^{n}(M)=M$ for any indecomposable $M$ in $\underline{F k}$.

Remark 3.16. The quivers of the cluster-tilting objects that are given by Scott's quadrilateral arrangements have the following shapes. If we write $T=T^{\prime} \oplus P_{1} \oplus P_{2} \oplus \cdots \oplus P_{n}$ where the $P_{i}$ are projective-injective, then the quiver of the endomorphism algebra of $T^{\prime}$, for $T^{\prime}$ arising from the quadrilateral arrangement is as on the left, with $k-1$ rows and $n-k-1$ columns. It can be mutated to the quiver on the right:


These quivers are also appear in Keller's work [Kel13] on the periodicity conjecture. In terms of these quivers, $\mathcal{F}_{k, n}$ has finite representation type if and only if it has a cluster-tilting object $T=T^{\prime} \oplus\left(\oplus_{i} P_{i}\right)$ such that the quiver of the endomorphism algebra of $T^{\prime}$ is a linear orientation of $\mathrm{A}_{n}$ or it is formed by 2,4 or 6 oriented triangles forming one rectangle with side lengths 2 and 2,3 or 4 .

### 3.6 Root systems associated to $\mathcal{F}_{k, n}$

Recall that $k \leq \frac{n}{2}$. The main references for this section are [JKS16, §8] and [BBGE19, §2].
Consider the graph
with nodes $1,2, \ldots, n-1$ on the horizontal line and node $n$ branching off from node $k$. To this graph, a root system $\Phi_{k, n}$ is associated, each node corresponds to a simple root and edges indicate simple roots which can be added. The root system of $J_{k, n}$ has simple roots $\alpha_{i}:=-e_{i}+e_{i+1}$ for $i=1, \ldots, n-1$ and $\beta=e_{1}+\cdots+e_{k}$ for $e_{1}, \ldots, e_{n}$ the standard basis vectors of $\mathbb{C}^{n}$. Note that if $k=2, J_{2, n}$ is a Dynkin diagram of type $\mathrm{D}_{n}$. If $k=3$ and $n \in\{6,7,8\}$, the graph $J_{3, n}$ is a Dynkin diagram of type $\mathrm{E}_{6}, \mathrm{E}_{7}$ or $\mathrm{E}_{8}$, respectively.
Let $\mathbb{Z}^{n}(k):=\left\{\underline{x} \in \mathbb{Z}^{n} \mid k\right.$ divides $\left.\sum_{i} x_{i}\right\}$. The root system $\Phi_{k, n}$ of $J_{k, n}$ can be identified with $\mathbb{Z}^{n}(k)$ via $\alpha_{i} \leftrightarrow-e_{i}+e_{i+1}$, for $i \leq n-1$ and $\beta \leftrightarrow e_{1}+\cdots+e_{k}$.
Define $q: \mathbb{Z}^{n}(k) \rightarrow \mathbb{Z}$ to be $q(\underline{x})=\sum_{i=1}^{n} x_{i}^{2}+\frac{2-k}{k^{2}}\left(\sum_{i=1}^{n} x_{i}\right)^{2}$.
Then the roots for $J_{k, n}$ correspond to the vectors $\underline{a}$ of $\mathbb{Z}^{n}(k)$ with $q(\underline{a}) \leq 2$. The vectors with $q(\underline{a})=2$ are the real roots, the vectors with $q(\underline{a})<2$ are imaginary roots. The degree of a root $\gamma$ is its coefficent at $\beta$ : If $\gamma=\sum_{i=1}^{n-1} m_{i} \alpha_{i}+m \beta, m_{i}, m \in \mathbb{Z}$, then $\operatorname{deg} \gamma=m$.
In Section 8, JKS16] define a map ind $\mathcal{F}_{k, n} \xrightarrow{\varphi} \Phi_{k, n}$. The map $\varphi$ associates to each indecomposable a positive root of $\Phi_{k, n}$.

Let $M$ be an indecomposable rank $d$ module. Assume that $M$ has a filtration by rank 1 modules, $M=\frac{\frac{\mathbb{M}_{I_{1}}}{\vdots}}{\frac{\mathbb{M}_{I_{d}}}{}}$ for $k$-subsets $I_{1}, \ldots, I_{d}$. For $i=1, \ldots, n$ let $a_{i}$ be the multiplicity of $i$ in $I_{1} \cup \cdots \cup I_{d}$. We
associate to $M$ the vector $\underline{a}(M)=\left(a_{1}, \ldots, a_{n}\right)$. Then $\varphi(M)$ is defined to be the root corresponding to $\left(a_{1}, \ldots, a_{n}\right) \in \mathbb{Z}^{n}(k)$.

Example 3.17. Let $n=6, k=3$. So $J_{3,6}$ is as follows:

$$
\alpha_{1}-\alpha_{2}-\stackrel{\alpha_{3}}{\alpha_{3}}-\alpha_{4}-\alpha_{5}
$$

Let $I=\{1,3,5\}$ and $J=\{2,4,6\}$.

1. $M:=\mathbb{M}_{I}: \underline{a}=(1,0,1,0,1,0)$ and $\varphi(M)=\beta+\alpha_{2}+\alpha_{3}+\alpha_{4}$.
2. Let $M=\mathbb{M}_{I} / \mathbb{M}_{J}$. Then $\underline{a}=(1,1,1,1,1,1)$ which corresponds to $2 \beta+\alpha_{1}+2 \alpha_{2}+3 \alpha_{3}+2 \alpha_{4}+\alpha_{5}$, the highest root for $\mathrm{E}_{6}$.

Exercise 12. Compute $\varphi(M)$ for $M=\mathbb{M}_{J}$ and for $M=\mathbb{M}_{J} / \mathbb{M}_{I}$ from Exercise 3.17
Remark 3.18. Let $M$ be indecomposable with filtration $M=\mathbb{M}_{I_{1}} / \mathbb{M}_{I_{2}} / \ldots / \mathbb{M}_{I_{d}}$. Then one observes that $\varphi(M)$ is a root of degree $d$.

Question 1. What is the connection between indecomposable rank $r$-modules in $\mathcal{F}_{k, n}$ and roots for $J_{k, n}$ ? What is the connection between rigid indecomposables in $\mathcal{F}_{k, n}$ and real roots for $J_{i, n}$

For $r=1$, there is a bijection

$$
\{\text { indecomposable rank 1-modules }\} / \sim \quad \stackrel{1: 1}{\longleftrightarrow} \quad\left\{\text { real roots for } J_{k, n} \text { of degree } 1\right\}
$$

In finite types, one finds

$$
\{\text { indecomposable rank } r \text {-modules }\} / \sim \quad \stackrel{r: 1}{\longleftrightarrow} \quad\left\{\text { real roots for } J_{k, n} \text { of degree } r\right\}
$$

as in these types, the higher rank modules "cycle": Let $k=3$ and $n \in\{6,7,8\}$, then the indecomposables are all of rank $\leq 3$. And $M=\mathbb{M}_{I} / \mathbb{M}_{J}$ is an indecomposable rank 2 module if and only if $\mathbb{M}_{J} / \mathbb{M}_{I}$ is indecomposable. Furthermore, $\varphi\left(\mathbb{M}_{I} / \mathbb{M}_{J}\right)=\varphi\left(\mathbb{M}_{J} / \mathbb{M}_{I}\right)$, cf. Exercise 12 for $n=6$. For the rank 3 modules, $n=8$, we have $\mathbb{M}_{I} / \mathbb{M}_{J} / \mathbb{M}_{L}$ is indecomposable if and only if $\mathbb{M}_{J} / \mathbb{M}_{L} / \mathbb{M}_{I}$ is indecomposable if and only if $\mathbb{M}_{L} / \mathbb{M}_{I} / \mathbb{M}_{J}$ is indecomposable, all with the same root in $\Phi_{3,8}$.

The first cases that are not fully understood are the tame cases $\mathcal{F}_{3,9}$ and $\mathcal{F}_{4,8}$, cf. Theorem 3.12. It is known that in these cases every real root of degree 2 yields exactly two rigid indecomposable rank 2 modules. But there are indecomposable rank 2 modules in $\mathcal{F}_{4,8}$ whose root is an imaginary degree 2 root. See Example 3.19 and Exercise 13

Example 3.19. Let $n=8, k=4, I=\{2,5,6,8\}$ and $J=\{1,3,4,7\}$. The lattice diagram of the indecomposable $\mathbb{M}_{I} / \mathbb{M}_{J}$ is here - vertices indicated with $\bullet$ correspond to one-dimensional vector spaces and vertices indicated with $\diamond$ correspond to two-dimensional vector spaces and arrows $=>$ to maps between two-dimensional vector spaces. The module $\mathbb{M}_{J}$ is seen as the submodule on the boxed vertices. One can check that the associated root is imaginary.


Note that $\mathbb{M}_{J} / \mathbb{M}_{I}$ is not indecomposable.
Exercise 13. Find $\varphi\left(\mathbb{M}_{I} / \mathbb{M}_{J}\right)$ for $I, J$ as in Example 3.19, Compute $q\left(\underline{a}\left(\mathbb{M}_{I} / \mathbb{M}_{J}\right)\right.$.

### 3.7 Rank 2 modules

Let $I$ and $J$ be two $k$-subsets. Assume that $I$ and $J$ are strictly 3-interlacing, i.e. that $|I \backslash J|=|J \backslash I|=3$ and that the non-common elements of $I$ and $J$ interlace. We want to define a rank 2 module $\mathbb{M}(I, J)$ in similar way as rank 1 modules are defined. Let $V_{i}:=\mathbb{C}[|t|] \oplus \mathbb{C}[|t|], i=1, \ldots, n$. We will need to say how $x_{i}, y_{i}$ act. For this, define matrices

$$
\begin{aligned}
A_{1}:=\left(\begin{array}{cc}
t & -2 \\
0 & 1
\end{array}\right) & B_{1}:=\left(\begin{array}{ll}
t & 0 \\
0 & 1
\end{array}\right)
\end{aligned} \quad C_{1}:=\left(\begin{array}{cc}
t & -1 \\
0 & 1
\end{array}\right) \quad D_{1}:=\left(\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right) .
$$

Note that these are all matrix factorisations of $\left(\begin{array}{cc}t & 0 \\ 0 & t\end{array}\right): A_{1} A_{2}=B_{1} B_{2}=C_{1} C_{2}=D_{1} D_{2}=\left(\begin{array}{cc}t & 0 \\ 0 & t\end{array}\right)$.
Example 3.20. Let $k=3, n=6, I=\{1,3,5\}$ and $J=\{2,4,6\}$. We define $\mathbb{M}(I, J)$ as follows: The vertices of $\Gamma_{n}$ have the vector spaces $V_{i}$. The maps $x_{i}, y_{i}$ are:

$$
x_{i}: V_{i-1} \rightarrow V_{i} \text { acts as }\left\{\begin{array} { l l } 
{ A _ { 1 } } & { \text { if } i = 1 } \\
{ B _ { 2 } } & { \text { if } i = 2 } \\
{ B _ { 1 } } & { \text { if } i = 3 } \\
{ C _ { 2 } } & { \text { if } i = 4 } \\
{ C _ { 1 } } & { \text { if } i = 5 } \\
{ A _ { 2 } } & { \text { if } i = 6 }
\end{array} \quad y _ { i } : V _ { i } \rightarrow V _ { i - 1 } \text { acts as } \left\{\begin{array}{ll}
A_{2} & \text { if } i=1 \\
B_{1} & \text { if } i=2 \\
B_{2} & \text { if } i=3 \\
C_{1} & \text { if } i=4 \\
C_{2} & \text { if } i=5 \\
A_{1} & \text { if } i=6
\end{array}\right.\right.
$$

We can define a rank 2 module in $\mathcal{F}_{k, n}$ more generally. Let $I$ and $J$ be strictly 3 -interlacing, write $I \backslash J$ as $\left\{a_{1}, a_{2}, a_{3}\right\}$ and $J \backslash I=\left\{b_{1}, b_{2}, b_{3}\right\}$ so that $1 \leq a_{1}<b_{1}<a_{2}<b_{2}<a_{3}<b_{3} \leq n$. For the arrows incident with the $a_{i} s$ and the $b_{i} \mathrm{~s}$, we use the construction from Example 3.20. For all other arrows, we use the maps $D_{1}, D_{2}$.

Definition 3.21. Let $I, J$ be strictly 3 -interlacing $k$-subsets of $[n]$. At the vertices of $\Gamma_{n}, \mathbb{M}(I, J)$ has the $V_{1}, \ldots, V_{n}$. We define the maps $x_{i}, y_{i}$ as follows:

$$
x_{i}: V_{i-1} \rightarrow V_{i} \text { acts as }\left\{\begin{array} { l l } 
{ A _ { 1 } } & { \text { if } i = a _ { 1 } } \\
{ B _ { 2 } } & { \text { if } i = b _ { 1 } } \\
{ B _ { 1 } } & { \text { if } i = a _ { 2 } } \\
{ C _ { 2 } } & { \text { if } i = b _ { 2 } } \\
{ C _ { 1 } } & { \text { if } i = a _ { 3 } } \\
{ A _ { 2 } } & { \text { if } i = b _ { 3 } } \\
{ D _ { 1 } } & { \text { if } i \in I \cap J } \\
{ D _ { 2 } } & { \text { if } i \in I ^ { c } \cap J ^ { c } }
\end{array} \quad y _ { i } : V _ { i } \rightarrow V _ { i - 1 } \text { acts as } \left\{\begin{array}{ll}
A_{2} & \text { if } i=a_{1} \\
B_{1} & \text { if } i=b_{1} \\
B_{2} & \text { if } i=a_{2} \\
C_{1} & \text { if } i=b_{2} \\
C_{2} & \text { if } i=a_{3} \\
A_{1} & \text { if } i=b_{3} \\
D_{2} & \text { if } i \in I \cap J \\
D_{1} & \text { if } i \in I^{c} \cap J^{c}
\end{array}\right.\right.
$$

Exercise 14. Check that $\mathbb{M}(I, J) \in \mathcal{F}_{k, n}$. For this, check that $x y=y x$ and $x^{k}=y^{n-k}$ at all vertices (hence is a $B$-module) and that $\mathbb{M}(I, J)$ is free over the centre.

Question 2. When is $\mathbb{M}(I, J)$ indecomposable?
It is known that if a rank two module with filtration $\mathbb{M}_{I} / \mathbb{M}_{J}$ is indecomposable then $I$ and $J$ have to be strictly 3-interlacing ([BBGE19] $)$. We claim that the converse is also true.

Proposition 3.22. Let $I, J$ be strictly 3-interlacing. Then $\mathbb{M}(I, J)$ is indecomposable.

Idea of proof. Recall that a module is indecomposable if and only if its endomorphism ring is local and this is the case if and only if any idempotent endomorphism of the module is 0 or the identity.
(1) First consider $n=6$. Any endomorphism of $\mathbb{M}(I, J)$ is of the form $\varphi=\left(\varphi_{i}\right)_{1 \leq i \leq 6}$ with $\varphi_{i}: V_{i} \rightarrow V_{i}$. One computes that $\varphi$ is as follows:

$$
\left(\varphi_{1},\left(\begin{array}{ll}
a & b \\
c t & d
\end{array}\right), \varphi_{1},\left(\begin{array}{cc}
a+c & b+(d-a-c) t^{-1} \\
c t & d-c
\end{array}\right), \varphi_{1},\left(\begin{array}{cc}
a+2 c & b+2(d-a-2 c) t^{-1} \\
c t & d-2 c
\end{array}\right)\right)
$$

for $\varphi_{1}=\left(\begin{array}{ll}a & b t \\ c & d\end{array}\right)$, with $a, b, c, d \in \mathbb{C}[[t]]$ and where $t \mid c$ and $t \mid(a-d)$. To see this, use the relations $x_{i} \varphi_{i+1}=\varphi_{i} x_{i}$ for $i=1, \ldots, 6$.
(2) Now one shows that if $\varphi$ is an idempotent endomorphism of $\mathbb{M}(I, J)$, then $b=c=0$ and $a=d=0$ or $a=d=1$. So any idempotent endomorphism of $\mathbb{M}(I, J)$ is the identity or 0 . Hence $\mathbb{M}(I, J)$ is indecomposable in case $n=6$.

In more detail: we consider $\varphi_{1}^{2}=\varphi_{1}$ :

$$
\varphi_{1}^{2}=\left(\begin{array}{cc}
a^{2}+b c t & (a+d) b \\
(a+d) c t & d^{2}+b c t
\end{array}\right)=\left(\begin{array}{cc}
a & b \\
c t & d
\end{array}\right)
$$

The equations $a^{2}+b c t=a \quad$ and $\quad d^{2}+b c t=d$ on the diagonal entries give $a-a^{2}=d-d^{2}$, i.e. $a-d=a^{2}-d^{2}=(a-d)(a+d)$ and hence $a=d$ or $a+d=1$. The equations also show that $t$ divides $a(1-a)$ and that $t$ divides $d(1-d)$.
Assume first $a=d$. If $b \neq 0$, we get $a=\frac{1}{2}$ but then $b c t=a-a^{2}=-\frac{1}{4}$, impossible. Analogously for $c \neq 0$. Thus $b=c=0$ and $a=d=0$ or $a=d=1$, the two trivial cases.

So assume $a \neq d$, so $d=1-a$. Combining $t \mid a(1-a)$ with the fact that $t$ divides $a-d=2 a-1$ implies to $t \mid 1$, a contradiction.
(3) In the general case, let $\varphi=\left(\varphi_{i}\right)_{1 \leq \leq n}$. One checks that for for all $i \in I \cap J \cup I^{c} \cap J^{c}, \varphi_{i}=\varphi_{i+1}$. Hence this reduces to the case $n=6$.

Question 3. How does a construction of indecomposable rank 3 modules look like?

### 3.8 Friezes from $\mathcal{F}_{k, n}$

The main reference for this Section is [ $\left.\mathrm{BFG}^{+} 18\right]$. Let $\mathcal{F}_{k, n}$ be of finite type.
Definition 3.23. A mesh frieze $M_{k, n}$ for $\mathcal{F}_{k, n}$ is a collection of positive integers, one for each indecomposable of $\mathcal{F}_{k, n}$ (up to isomorphism) such that $M_{k, n}(P)=1$ for every indecomposable projective $P$ and such that all mesh relations evaluate to 1 . In order words: whenever we have an AuslanderReiten sequence $A \rightarrow \oplus_{i} B_{i} \rightarrow C$ with $B_{i}$ indecomposable, $M_{k, n}(A) \cdot M_{k, n}(C)=\prod_{i} M_{k, n}\left(B_{i}\right)+1$.

Remark 3.24. A mesh frieze $M_{2, n}$ is a Conway-Coxeter frieze, also called $\mathrm{SL}_{2}$-frieze. Such friezes are in bijection with triangulations of polygons ([CC73a, CC73b]) and thus arise from specialising a cluster-tilting object in a cluster category $\mathcal{F}_{2, n}$ of type $\mathrm{A}_{n-3}$ to 1 .

Remark 3.25. The friezes of Dynkin types of [ARS10] correspond to our mesh friezes for $\mathcal{F}_{k, n}$ in types $A, D_{4}, E_{6}, E_{8}$.

Definition 3.26. An $\mathrm{SL}_{3}$-frieze is an array starting and ending with $k-1$ rows of 0 s and with finitely many rows of positive integers in between, arranged as below, and such that each $3 \times 3$ matrix has determinant 1. The width of an $\mathrm{SL}_{3}$-frieze is the number of rows of positive integers between the two rows of 1 s .


Proposition 3.27. Let $n \in\{6,7,8\}$. There is a bijection

$$
\left\{\text { mesh friezes for } \mathcal{F}_{3, n}\right\} \stackrel{1: 1}{\longleftrightarrow}\left\{S L_{3} \text {-friezes of width } n-4\right\} \text {. }
$$

For $n=6$, this is in [MG15], for $n=7,8$ in $\mathrm{BFG}^{+} 18$ ].
By the above result, in order to study mesh friezes for $\mathcal{F}_{3, n}$ it is equivalent to study $\mathrm{SL}_{3}$-friezes of width $n-4$. We will use the two notions interchangeably.

Most of the mesh friezes for $k=3$ arise from specialising a cluster-tilting object in $\mathcal{F}_{3, n}$ to 1 , but there are also other mesh friezes $M_{3, n}$, see Remark 3.28. If a mesh frieze arises from specialising a cluster-tilting object to 1 , it is called unitary. Otherwise, the mesh frieze is non-unitary.

Remark 3.28. The number of mesh friezes for $\mathcal{F}_{3, n}$ are not known for $n=8$ :

| $n$ | 6 | 7 | 8 |
| :--- | ---: | ---: | :---: |
| unitary | 50 | 833 | 25080 |
| non-unitary | 1 | 35 | $1872 ?$ |
| all | 51 | 868 | $26952 ?$ |

The non-unitary $\mathrm{SL}_{3}$ frieze of width 2 arises from specialising the non projective-injective summands of a cluster-tilting object to 2 's. Cuntz-Plamondon prove in an appendix to [ $\left.\mathrm{BFG}^{+} 18\right]$ that the number of non-unitary $\mathrm{SL}_{3}$-friezes of width 3 is 35 . These arise from specialising the non projective-injective summands of a cluster-tilting object to four 2's and two 1's. See Example 3.30

Remark 3.29. We can use lyama-Yoshino reduction to show that all known non-unitary mesh friezes arise from non-unitary friezes of type $D_{4}$ (i.e. from the non-unitary mesh frieze for $\mathcal{F}_{3,6}$ or from type $\mathrm{D}_{6}$ (these are in [FP16]) or from specialising the non projective-injective summands of a cluster-tilting object of $\mathcal{F}_{3,8}$ to 3 's. See Example 3.30

Example 3.30. The mesh frieze for $\mathcal{F}_{3,6}$ arises from (two ways of) specialising a cluster-tilting object to 2 's. The quiver of the endomorphism algebras (of the non projective-injective summands) of these two ways are belows. For $\mathcal{F}_{3,8}$ we obtain 4 non-unitary mesh friezes by specialising (the non projectiveinjective summands of) a cluster-tilting object to 3 's. The quiver of the endomorphism algebra is on the right. In these mesh friezes, all entries are $>1$.


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