CIMPA EXERCISES

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My hope for this course is not to lecture everything that is in the notes I have shared. This is way too much to cover in 4.5 hours of lectures. Instead, I want to give you an overview of the results, and set you up to read the book at your own pace. These exercises will allow you to start that process during the two tutorial sessions, and I will spend lecture explaining how the different pieces fit together.

1. Classical representation theory of SL_2

Some of you will have already worked through the representation theory of SL_2, some of you will not yet have done so. To put us on the same page, the first exercise has you check your understanding of Chapter 1 of the notes. These exercises are all solved in the first chapter of the notes. Please think through them as long as they are useful practice, and consult the notes when you are satisfied and want to check your understanding.

If all solutions are familiar to you already, then please instead skip to Section 2, which includes repeating the exercises for the quantum group.

(1) **Classify the irreducible finite-dimensional representations of SL_2.**
   - (a) Choose an eigenvector v for H of largest real component λ. This is called the highest weight of the representation.
   - (b) Show that $Ev = 0$, and that $F^k v = 0$ for some k.
   - (c) Show by induction that $HF^m v = (\lambda - 2m)F^m v$ for all m.
   - (d) Show that $EF^m v = (\lambda + 1 - m)F^{m-1}v$, for all m.
   - (e) Conclude that λ is an integer (!). Denote by $V(\lambda)$ the unique up to isomorphism finite-dimensional representation with highest weight λ which was constructed above. Show that it has dimension $\lambda + 1$.

   Conclude that for each integer λ there exists a unique irreducible SL_2-representation $V(\lambda)$ with highest weight k, and that every irreducible finite-dimensional SL_2-representation is of this form.

   Remark 1. *For a general simple group G, there is a similar classification of finite-dimensional irreducible representations $V(\lambda)$ according to a highest weight λ. This is no longer an integer, but rather a tuple of r integers indexing the simultaneous eigenvalues for a distinguished basis $H_1, \ldots, H_r$ of generators of the Cartan subalgebra T of G (or more invariantly an element of a "weight lattice", $\mathrm{Hom}(T, \mathbb{C}^\times)$).*

(2) **Compute the character of a finite-dimensional representation.** Given a finite-dimensional SL_2-representation V and an integer μ, let V_μ denote the subspace of V on

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which H acts with weight μ. Let $\mathrm{ch}(V) = \sum x^k \dim(V_k)$. Show that

$$\mathrm{ch}(V(\lambda)) = x^\lambda + x^{\lambda-2} + \cdots + x^{2-\lambda} + x^{-\lambda} = \frac{x^{\lambda+1} - x^{-\lambda-1}}{x - x^{-1}},$$

and in particular that its dimension is $\lambda + 1$.

Remark 2. *For a general group G with irreducible module $V(\lambda)$, the analogous formula for its character and dimension are called the Weyl character and Weyl dimension formulas, respectively.*

(3) **Determine the center of $U(\mathfrak{sl}_2)$.**

(a) Show by direct computation that the element $C = EF + FE + \frac{1}{2}H^2$ is central.
(b) For now, assume the PBW theorem: a basis for $U(\mathfrak{sl}_2)$ consists of all ordered monomials, $E^k H^l F^m$, for non-negative integers k, l, m.
(c) Conclude that the powers C^k for all k are linearly independent.
(d) Show that the associated graded of $U(\mathfrak{sl}_2)$ is isomorphic to

$$\mathrm{Sym}(V) = \bigoplus_k \mathrm{Sym}^k(V(2)).$$

(e) Consider the action of H on $\mathrm{Sym}(V(2))$. Show its graded character is given by the power series expansion in t of

$$\frac{1}{(1 - tx^{-2})(1 - t)(1 - tx^2)},$$

where the coefficient of t^k is the character of $\mathrm{Sym}^k V(2)$.

Conclude that the graded dimension of $(\mathrm{Sym}(V(2)))^{\mathfrak{sl}_2}$ is $\frac{1}{1-t^2}$, and hence that the Casimir C generates the centre of $U(\mathfrak{sl}_2)$. (Hint: by the character formula above, you only need to compute the x coefficient of

$$\frac{x - x^{-1}}{(1 - tx^{-2})(1 - t)(1 - tx^2)},$$

as a power series in t.

Remark 3. *For a general simple group G, we have the Chevalley isomorphism,*

$$Z(U(\mathfrak{g})) \cong U(\mathfrak{g})^{\mathfrak{g}} \cong \mathrm{Sym}(\mathfrak{h})^W,$$

which identifies the center of the universal enveloping algebra with the W-invariant polynomials functions on $\mathfrak{h}^*$.

(4) **Show that every finite-dimensional $U(\mathfrak{sl}_2)$-representation is a direct sum of irreducible representations.**

(a) Show that C acts on $V(\lambda)$ by the scalar $c_\lambda = \frac{\lambda(\lambda+2)}{2}$.
(b) Let V be an arbitrary representation. Let $V[\lambda]$ denote the subspace on which C acts by c_λ. Give a decomposition,

$$V = \bigoplus_{k=0}^{\infty} V[\lambda].$$

(c) Show that the category is semisimple, i.e. that every representation is a direct sum of simples. (This is hard, I suggest reading the notes together in a group).

Remark 4. *This theorem holds as stated for any simple algebraic group G.*

(5) Produce an isomorphism $\mathcal{O}(\mathrm{SL}_2)^{\mathrm{SL}_2} \cong \mathcal{O}(H)^{\mathbb{Z}/2}$, where $\mathbb{Z}/2$ acts on H by inversion. Hint: while it's not quite true that every matrix can be diagonalised, show that every matrix sits arbitrarily close to a diagonal matrix, so that a G-invariant algebraic function on SL_2 is uniquely determined by its values on the diagonalisable matrices.

Remark 5. *This statement and line of proof generalises to an arbitrary reductive group, giving:*

$$\mathcal{O}(G)^G \cong \mathcal{O}(H)^W.$$

2. Quantum representation theory of SL_2

These exercises are intended to introduce you the students to the representation theory of the quantum group. The presentation is somewhat non-standard. So I hope it will be interesting even if you've already seen another construction of the quantum group. The main idea as you will learn from the exercises is that the quantum group behaves very similarly to the classical universal enveloping algebra.

Recall the quantum group $\mathcal{O}_q(SL_2)$. It has generators a, b, c, d, with commutation relations:

$$\begin{aligned} da &= ad & ba &= ab + (1 - q^{-2})bd \\ db &= q^2 bd & ca &= ac + (q^{-4} - q^{-2})cd \\ dc &= q^{-2} cd & cb &= q^2 bc + (1 - q^{-2})(1 - d^2) \end{aligned}$$

and the quantum determinant relation,

$$ad - q^2 bc = 1.$$

(1) **Show that the algebra $\mathcal{O}_q(G)$ is a deformation of both $\mathcal{O}(SL_2)$ and $U(\mathfrak{sl}_2)$.**

(a) Show that at $q = 1$ the relations become those of $\mathcal{O}(\mathrm{SL}_2)$ (this is easy).

(b) Consider the $\mathbb{C}[q, q^{-1}]$-subalgebra $\bar{\mathrm{U}}$ generated by

$$\bar{E} = \frac{b}{q - q^{-1}}, \qquad \bar{F} = \frac{c}{q - q^{-1}}, \qquad \bar{H} = (1 - d^2).$$

Show that at $q = 1$ this subalgebra becomes isomorphic to $U(\mathfrak{sl}_2)$.

Remark 6. *Quantum groups can be defined in similar fashion for any simple group, and they have the same property, that they can be degenerated to either the coordinate algebra of the group or to the universal enveloping algebra.*

Remark 7. *The presentation given here for the quantum group is non-standard, and usually another algebra called $U_q(\mathfrak{g})$ is dubbed the quantum group, or more accurately the quantum universal enveloping algebra. The algebra $U_q(\mathfrak{sl}_2)$ is presented with generators $E, F, K^{\pm 1}$ subject to the* quantum Serre *relations:*

$$KEK^{-1} = q^2 E, \quad KFK^{-1} = q^{-2} F, \quad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}. \tag{1}$$

Consider the algebra $\mathcal{O}_q(\mathrm{SL}_2)[d^{-1}]$, and let

$$E = \frac{d^{-1}b}{q-q^{-1}}, \quad F = \frac{c}{q-q^{-1}}, \quad K = d.$$

(2) **Prove that $\mathcal{O}_q(\mathrm{SL}_2)[d^{-1}]$ is generated by E, F, K, K^{-1}, subject to the quantum Serre relations (1).**

Remark 8. *This tells us that $U_q(\mathfrak{g})$ is not a quantization of the group G but of a 2-fold cover open subset G° known as the "big cell". This cover is known the Poisson-Lie dual group $G^* := B_+ \times_T B_-$.*

(3) **Prove that** $\mathrm{tr}_q = a + q^{-2}d$ **is central**, and that under the above isomorphism, maps to a multiple of the *quantum Caismir element*,

$$C_q = EF + \frac{Kq^{-1} + K^{-1}q}{(q-q^{-1})^2}.$$

(4) **Repeat exercises (1)-(4) from Section 1, but with $\mathcal{O}_q(G)$ in place of $U(\mathfrak{sl}_2)$.**

3. Warm-ups in algebraic geometry

These exercises are intended to give you some sense of some of the ideas in algebraic geometry even if you've never seen it before. As a bonus, these are really the only algebraic varieties we will consider in this course, so if you can understand how these examples work, you can do geometric representation theory for SL_2!

(1) **Show that $\mathbb{C}\backslash S$ is affine, whenever S is a finite set.** (Hint: the relation $x \cdot y = 1$ spells out a curve in $\mathbb{C}^2$. Why is this relevant?)

(2) **Show however that $\mathbb{C}^2\backslash\{\mathbf{0}\}$ is not affine.** Here $\mathbf{0} = (0,0) \in \mathbb{C}^2$. (Hint: what is the algebra of polynomial functions on it, and why is that a problem?)

(3) **Show that $\mathbb{CP}^1$ is not affine.** (Hint: what is the algebra of polynomial functions on it, and why is that a problem?)

(4) **Determine the categories $\mathrm{Coh}_{\{\mathbf{0}\}}(\mathbb{C}^2)$ and $\mathrm{Coh}_{\mathbb{C}^2\backslash\{\mathbf{0}\}}(\mathbb{C}^2)$,** of coherent sheaves supported at $\{\mathbf{0}\}$ in $\mathbb{C}^2$, and away from $\{\mathbf{0}\}$, respectively.

The *support* of a module M over a commutative algebra A is the set of maximal ideals I such that $M/IM \neq 0$. The support of a sheaf $\mathcal{F}$ on some variety X is the union of its supports as defined above with respect to some open cover by affine varieties.

(a) Show that a coherent sheaf $\mathcal{F}$ on $\mathbb{C}^2$ (regarded as a $\mathbb{C}[x,y]$-module M) has support $\{\mathbf{0}\}$ if, and only if, for all $m \in M$, there exists a, b such that $x^a m = y^b n = 0$. These are called torsion sheaves, and denoted Torsion.

(b) Show that torsion sheaves form a Serre subcategory: they are an abelian subcategory and are moreover closed under short exact sequences.

(c) Produce equivalences of categories,

$$\mathrm{Coh}(\mathbb{C}^2\backslash\{\mathbf{0}\}) \simeq \mathrm{Coh}_{\mathbb{C}^2\backslash\{\mathbf{0}\}}(\mathbb{C}^2) \simeq \mathbb{C}[x,y]\text{-mod}/\text{Torsion}.$$

4. Peter-Weyl and Borel-Weil

Okay, now let's apply our insights to prove two out of the three theorems in this course: the Peter-Weyl theorem, and its easy corollary, the Borel-Weil theorem.

Given a finite-dimensional representation V of $U(\mathfrak{sl}_2)$, and elements $v \in V$ and $f \in V^*$, define the matrix coefficient $c_{f,v}$ to be the linear functional,

$$\begin{aligned} c_{f,v} :& U(\mathfrak{sl}_2) \to \mathbb{C}, \\ & X \mapsto f(Xv). \end{aligned}$$

Let $\mathcal{O}$ denote the subspace of $U(\mathfrak{sl}_2)$ spanned by the $c_{f,v}$ for all V, for all $v \in V$ and $f \in V^*$. His has an algebra structure via $c_{f,v} \cdot c_{g,w} = c_{g\otimes f, v\otimes w}$. We will now produce an isomorphism of algebra $\mathcal{O} \cong \mathcal{O}(\mathrm{SL}_2)$.

We obtain a homomorphism $V^*\otimes V \to \mathcal{O}(\mathrm{SL}_2)$ of $U(\mathfrak{sl}_2)\otimes U(\mathfrak{sl}_2)$-modules by sending $f\otimes v \mapsto c_{f,v}$.

(1) **Construct an isomorphism between $\mathcal{O}(\mathrm{SL}_2)$ and the abstractly defined algebra $\mathcal{O}$ of matrix coefficients on $U(\mathfrak{sl}_2)$.**

Denote the standard basis of $V(1)$ by v_1, v_2, and the dual basis f^1, f^2. Abbreviate by c^i_j the matrix coeffcient c_{f^i,v_j}.

(a) Show that there exists a unique homomorphism:

$$\phi : \mathbb{C}[a,b,c,d] \to \mathcal{O},$$

$$(a,b,c,d) \mapsto (c^1_1, c^1_2, c^2_1, c^2_2).$$

Hint: you need to check that matrix coefficients form a commutative algebra (why?).

(b) Show that the relations $c_{f,i_0(v)} = c_{\pi_0(f),v}$, for $f = v^1, v^2$ and $v = v_1, v_2$, reduce to the single relation $ad - bc = 1$ (this is kind of hard/detailed).

(c) The algebra $\mathcal{O}(SL_2) = \mathbb{C}[a,b,c,d]/\langle ad - bc - 1\rangle$ admits a filtration with generators a, b, c, d in degree one. Let F_i denote the ith filtration, and show that F_i/F_{i-1} has a basis:

$$\mathcal{B}_i = \{a^k d^l c^m \mid k+l+m = i\} \cup \{a^k d^l b^m \mid k+l+m = i\},$$

so that $\dim F_i/F_{i+1} = |\mathcal{B}_i| = 2\binom{i+2}{2} - (i+1) = (i+1)^2$.

(d) Show that ϕ is a map of filtered vector spaces, where[1]

$$F_i(\mathcal{O}) = \oplus_{k \leq i} V(k)^* \boxtimes V(k).$$

(e) Conclude that ϕ is injective. Using that $V(k)^* \boxtimes V(k)$ is irreducible, conclude that ϕ is also surjective, and thus an isomorphism of algebras.

You have proved the Peter-Weyl theorem: you have constructed an isomorphism,

$$\mathcal{O}(\mathrm{SL}_2) \cong \oplus_\lambda V(\lambda)^* \boxtimes V(\lambda)$$

(2) **Classify all line bundles on** $\mathbb{CP}^1$: Show that every line bundle is of the form $\mathcal{O}(\lambda)$ for some integer λ, in two different ways:

(a) A line bundle on $\mathbb{CP}^1$ is glued out of two line bundles on $\mathbb{C}$. Using that $\mathbb{C}[x]$ is a PID, it is possible to show that the only line bundle is the trivial one (you may assume this fact without proof). What are the possible transition maps on the intersection $\mathbb{C}^\times$?

(b) A line bundle on $\mathbb{CP}^1$ may also be understood as a $\mathbb{C}^\times$-equivariant vector bundle on $\mathbb{C}^2 \backslash \{\mathbf{0}\}$, or in other words a free, graded, $\mathbb{C}[x, y]$-module. Show that every such module is just a copy of $\mathbb{C}[x, y]$ shifted into some degree λ.

(3) **Describe the category** $\mathrm{Coh}(\mathbb{CP}^1)$ **explicitly**. Building on Exercise 4 (d), produce an equivalence of categories,

$$\mathrm{Coh}(\mathbb{CP}^1) \simeq (\text{graded } \mathbb{C}[x, y]\text{-modules})/\text{Torsion}$$

(4) **Prove that for** $\lambda \geq 0$**, we have** $\dim \Gamma(\mathbb{CP}^1, \mathcal{O}(\lambda)) = \lambda + 1$. Prove this in two different ways, according to the two different classifications of line bundles on $\mathbb{CP}^1$ given above.

(5) **Use the Peter-Weyl theorem to prove the Borel-Weil theorem.** Give an isomorphism, $\Gamma(\mathbb{CP}^1, \mathcal{O}(\lambda)) \cong V(\lambda)$. Here, we regard a line bundle on $\mathbb{CP}^1$ as a B-equivariant line bundle on G, so that $\Gamma(\mathbb{CP}^1, \mathcal{O}(\lambda))$ obtains a G-action by left multiplication.

[1] the symbol $\boxtimes$ denotes the usual tensor product $\otimes$ of vector spaces, but where the two copies of $U(\mathfrak{sl}_2)$ act independently on each factor.