This book is an introduction to geometric representation theory. What is geometric representation theory? It is hard to define exactly what it is as this subject is constantly growing in methods and scope. The main aim of this area is to approach representation theory which deals with symmetry and non-commutative structures by geometric methods (and also get insights on the geometry from the representation theory). Here by geometry we mean any local to global situation where one tries to understand complicated global structures by gluing them from simple local structures. The main example is the Beilinson-Bernstein localization theorem. This theorem essentially says that the representation theory of a semi-simple Lie algebra (such as $\mathfrak{sl}(n, \mathbb{C})$) is encoded in the geometry of its flag variety. This theorem enables the transfer of "hard" (global) problems about the universal enveloping algebra, to "easy" (local) problems in geometry. The Beilinson-Bernstein localization theorem has been extremely useful in solving problems in representation theory of semi-simple Lie algebras and in gaining deeper insight into the structure of representation theory as a whole. There are many more examples of geometric representation theory in action, from Deligne-Lusztig varieties to the geometric Langlands' program and categorification.

The focus of this book is the Beilinson-Bernstein localization theorem. It follows the advice of the great mathematician Israel M. Gelfand: we only cover the case of $\mathfrak{sl}_2$ (classical and quantum). This approach allows us to introduce many topics in a very concrete way without going into the general theory. Thus we cover the Peter-Weyl theorem, the Borel-Weil theorem, the Beilinson-Bernstein theorem and much more for both the classical and quantum case. Dealing with the quantum case allows us also to introduce many tools from non-commutative algebraic geometry and quantum groups. These topics are usually considered very advanced. To have a full understanding of them requires a good grasp of algebraic geometry, D-module theory, category theory, homological algebra and the theory of semi-simple Lie algebras. We think that by focusing on the simplest case of $\mathfrak{sl}_2$ the student can gain much insight and intuition into the subject. A good and deep understanding of $\mathfrak{sl}_2$ makes the general theory much simpler to learn and appreciate.

This book is based on a graduate lecture course given at MIT by the second author. We are grateful to the students taking that course for sharing their notes with us as we prepared this manuscript.

Contents

Introduction

In representation theory, the Lie algebra $\mathfrak{sl}_2 := \mathfrak{sl}_2(\mathbb{C})$ comprises the first and most important example of a semi-simple Lie algebra. In this introductory text, which grew out of a course taught by the first author, we will walk the reader through important concepts in geometric representation theory, as well as their quantum group analogues. Our focus is on developing concrete examples to illustrate the geometric notions discussed in the text. As such, we will restrict our attention almost exclusively to $\mathfrak{sl}_2$, giving more general definitions only when it is convenient or illustrative.

In Chapter 1, we show that the category of finite-dimensional $\mathfrak{sl}_2$-modules is a semi-simple abelian category; we prove this important fact in a way which will generalize most easily to the quantum setting in later chapters.

In Chapter 2, we introduce the formalisms of Hopf algebras and tensor categories. These capture the essential properties of algebraic groups, their representations, and their coordinate algebras, in a way that can be extended to the quantum setting.

In Chapter 3, we discuss the relation between geometry of various G-varieties and the representation theory of G. We discuss the Peter-Weyl theorem, and obtain as a corollary the Borel-Weil theorem. We define D-modules on $\mathbb{P}^1$, and we relate them to representations of $\mathfrak{sl}_2$: this is the first instance of the so-called Beilinson-Bernstein localization theorem.

In Chapter 4, we introduce the quantized universal enveloping algebra $U_q(\mathfrak{sl}_2)$, and extend the results of Chapter 1 to the quantum setting.

In Chapter 5, we explain the notion of a braided tensor category, a mild generalization of the notion of a symmetric tensor category. Braided tensor categories underlie the representation theory of $U_q(\mathfrak{sl}_2)$ in a way analogous to the role of symmetric tensor categories in the representation theory of $\mathfrak{sl}_2$.

In Chapter 6, we reproduce the results of Chapter 3 in the quantum setting. We have quantum analogs of each of the Peter-Weyl, Borel-Weil, and Beilinson-Bernstein theorems.

Throughout the text assume some passing familiarity with the theory of Lie algebras. Two excellent introductions are Humphreys [**?**] and Knapp [**?**].

CHAPTER 1

The first classical example: $\mathfrak{sl}_2$.

1. The Lie algebra $\mathfrak{sl}_2$

The Lie algebra $\mathfrak{sl}_2 := \mathfrak{sl}_2(\mathbb{C})$ consists of the traceless 2×2 matrices, with the standard Lie bracket:

$$[A, B] := AB - BA.$$

A standard and convenient presentation of $\mathfrak{sl}_2(\mathbb{C})$ is given as follows. We let:

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1}$$

Then $\mathfrak{sl}_2$ is spanned by E, F, and H, with commutators:

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H.$$

Recall that a representation of $\mathfrak{g}$ (equivalently, a $\mathfrak{g}$-module) is a vector space V, together with a Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathrm{End}(V)$. We will often omit ρ from notation, and write simply $x.v$ for $\rho(x).v$.

The finite-dimensional representations of $\mathfrak{sl}_2$ are sufficiently complicated to be interesting, yet can be completely understood by elementary means. In this chapter, we recall their classification. We begin with some examples:

Example 1.1. The defining representation. The Lie algebra $\mathfrak{sl}_2$ acts on $\mathbb{C}^2$ by matrix multiplication.

Example 1.2. The adjoint representation. Any Lie algebra $\mathfrak{g}$ acts on itself by $x.y := [x, y]$.

Example 1.3. Given any representation V of a Lie algebra $\mathfrak{g}$, its dual vector space V^* carries an action defined by $(X.f)(v) = f(-X.v)$. The corresponding representation is also denoted V^*.

Example 1.4. Given two representations V and W of $\mathfrak{g}$, the vector space $V \oplus W$ carries an action of $\mathfrak{g}$ defined by $x(v, w) := (xv, xw)$ for $(v, w) \in V \oplus W$, and $x \in \mathfrak{g}$. The corresponding representation is also denoted $V \oplus W$.

Example 1.5. Given two representations V and W, the vector space $V \otimes W$ carries an action of $\mathfrak{g}$ defined by $x(v \otimes w) = x(v) \otimes w + v \otimes x(w)$, for $v \otimes w \in V \otimes W$, and $x \in \mathfrak{g}$. The corresponding representation is also denoted $V \otimes W$.

As we will see in Chapter 2, these examples make the category of $\mathfrak{g}$-modules into an abelian tensor category with duals (see also [?]).

2. Irreducible finite-dimensional modules

Definition 2.1. Let V be an $\mathfrak{sl}_2$-module. A non-zero $v \in V$ is a *weight vector* of weight λ if $Hv = \lambda v$. A *highest weight vector* is a weight vector v of V such that $Ev = 0$. Denote by V_λ the subspace of weight vectors of weight λ.

Observe that commutation relations (1) imply $EV_\lambda \subset V_{\lambda+2}$, and $FV_\lambda \subset V_{\lambda-2}$.

Exercise 2.2. Prove that every finite dimensional $\mathfrak{sl}_2$ module has a highest weight vector.

It follows that any irreducible finite dimensional representation is generated by a highest weight vector; this fact will be the key to their classification.

Lemma 2.3. *Let V be a finite-dimensinal $\mathfrak{sl}_2$-module, and suppose there exists a highest weight vector v_0, of weight λ. Let $v_i := (1/i!)F^i(v_0)$ (by convention, $v_{-1} = 0$). Then we have that:*

$$Hv_i = (\lambda - 2i)v_i, \quad Fv_i = (i+1)v_{i+1}, \quad Ev_i = (\lambda - i + 1)v_{i-1}. \tag{2}$$

Proof. The first two relations are obvious, and the third is a straightforward computation:

$$\begin{aligned} iEv_i &= EFv_{i-1} = [E,F]v_{i-1} + FEv_{i-1} \\ &= Hv_{i-1} + FEv_{i-1} = (\lambda - 2i + 2)v_{i-1} + (\lambda - i + 2)Fv_{i-2} \\ &= (\lambda - 2i + 2)v_{i-1} + (i-1)(\lambda - i + 2)v_{i-1} = i(\lambda - i + 1)v_{i-1}. \end{aligned}$$

□

Theorem 2.4. *Let V be an irreducible finite dimensional $\mathfrak{sl}_2$-module. Then V has a unique (up to scalar) highest weight vector of weight $m := \dim V - 1$. Further, V decomposes as a direct sum of one dimensional weight spaces of weights $m, m-2, \ldots, 2-m, -m$.*

Proof. It follows from Lemma 2.3 that $\mathrm{span}\{v_i\}_{i\in\mathbb{N}}$ is a submodule of V, and thus all of V. We let $m \geq 0$ be maximal such that $v_m \neq 0$ (equivalently, v_{m+1} is the first which is zero). Then by the third equation of equation (2): $0 = Ev_{m+1} = (\lambda - m)v_m$. Therefore we see that $\lambda = m$, and that $dimV = m + 1$. Further, it is immediate that the three formulas (with $\lambda = m$) define a representation of $\mathfrak{sl}_2$ on a vector space of dimension $m + 1$, which we will denote $V(m)$. Any such representation is irreducible, as applying E to a vector w repeatedly will eventually yield a nonzero multiple of v_0, and thus w generates all of V. □

We note three important examples: firstly, the trivial representation is the weight zero irreducible. The defining representation of $\mathfrak{sl}_2$ on 2-space is the weight one irreducible. Finally, we note that the adjoint representation is three dimensional of highest weight 2, and this implies that $\mathfrak{sl}_2$ is a simple Lie algebra.

3. The universal enveloping algebra

The universal enveloping algebra $U(\mathfrak{g})$, of a Lie algebra $\mathfrak{g}$, is the quotient of the free associative algebra on the vector space $\mathfrak{g}$ (i.e. the tensor algebra $T(\mathfrak{g})$), by the commutator relations $a \otimes b - b \otimes a = [a, b]$. That is,

$$U(\mathfrak{g}) := T(\mathfrak{g})/\langle a \otimes b - b \otimes a - [a, b]\rangle.$$

The canonical inclusion $\mathfrak{g} \hookrightarrow T(V)$ induces a natural map $i : \mathfrak{g} \to U(\mathfrak{g})$. This gives rise to a functor U from Lie algebras to associative algebras. We also have a forgetful functor F from associative algebras to Lie algebras, given by defining $[a, b] := ab - ba$, and then forgetting the associative multiplication.

Remark 3.1. Actually, the PBW theorem implies that the map $i : \mathfrak{g} \to U(\mathfrak{g})$ is an inclusion, but this is not needed in what follows.

Proposition 3.2. *The functors (U, F) form an adjoint pair.*

Proof. We need an isomorphism $\phi : \mathrm{Hom}(U(\mathfrak{g}), A) \to \mathrm{Hom}(\mathfrak{g}, F(A))$. Given $f : U(\mathfrak{g}) \to A$, we define $\phi(f) = f \circ i$. It is easy to check that this gives the required isomorphism. □

By the adjointness above,a $\mathfrak{g}$-module is the same as an associative algebra homomorphism $\rho : U(g) \to \mathrm{End}(V)$. In other words, we have an equivalence of categories $\mathfrak{g}$-Mod $\sim U(\mathfrak{g})$-Mod. Thus we may view representation theory of Lie algebras as a sub-branch of representation theory of associative algebras, rather than something entirely new.

The universal enveloping algebra of $\mathfrak{sl}_2$ contains an important central element, which will feature in the next section.

Definition 3.3. The Casimir element, $C \in U(\mathfrak{sl}_2)$, is given by the formula:

$$C = EF + FE + \frac{H^2}{2}.$$

Claim 3.4. *C is a central element of $U(\mathfrak{sl}_2)$.*

PROOF. It suffices to show that C commutes with the generators E, F, H. We compute:

$$\begin{aligned}[E,C] &= [E,EF] + [E,FE] + [E,\frac{H^2}{2}] \\ &= [E,E]F + E[E,F] + [E,F]E + F[E,E] \\ &\quad + \frac{1}{2}([E,H]H + H[E,H]) \\ &= EH + HE - EH - HE = 0.\end{aligned}$$

The bracket $[C,F]$ is zero by a similar computation or by consideration of the automorphism switching E and F and taking H to $-H$.

Taking the bracket with H gives:

$$\begin{aligned}[H,C] &= [H,E]F + E[H,F] + [H,F]E + F[H,E] \\ &= 2EF - 2EF - 2FE + 2FE = 0,\end{aligned}$$

which proves the claim. □

4. Semisimplicity

Having classified irreducible finite dimensional representations, we now wish to extend this classification to all finite dimensional representations. This is accomplished by the following:

THEOREM 4.1. *The category of finite dimensional $\mathfrak{sl}_2$-modules is semisimple: any finite dimensional $\mathfrak{sl}_2$-module is projective and thus decomposes as a direct sum of simples.*

In the proof of the theorem, we will use the following characterization of semi-simplicity:

EXERCISE 4.2. Show that an abelian category is semi-simple if, and only if, for every object X the functor $Hom(X,-)$ is projective. Hint: for the "if" direction, consider an exact sequence $0 \to U \to V \to W \to 0$, and apply the functor $Hom(W,-)$ to produce the required splitting $W \to V$.

By the exercise, we need to show that, for any finite dimensional $\mathfrak{sl}_2$-module X, the functor $Hom_{\mathfrak{sl}_2}(X,-)$ is exact on finite dimensional modules. We have a natural isomorphism,

$$\phi : Hom_{\mathfrak{sl}_2}(V, W^* \otimes L) \xrightarrow{\sim} Hom_{\mathfrak{sl}_2}(V \otimes W, L).$$

$$f \mapsto \phi(f),$$

where $\phi(f)(v \otimes w) := \langle f(v), w\rangle$. Let I denote the trivial representation; we have a natural isomorphism, $X \cong I \otimes X$, for any X. Therefore we have natural isomorphisms:

$$Hom_{\mathfrak{sl}_2}(X, V) \cong Hom_{sl(2)}(I \otimes X, V) \cong Hom_{\mathfrak{sl}_2}(I, X^* \otimes V).$$

As these are all vector spaces, tensoring by X^* is an exact functor. So we see that to prove the claim it suffices to show that $Hom_{\mathfrak{sl}_2}(I, -)$ is exact.

A homomorphism from the trivial module into V is simply the choice of a vector v with the property that $xv = 0$ for all $x \in \mathfrak{sl}_2$. The set of all such v is a submodule of V, denoted $V^{\mathfrak{sl}_2}$, which is naturally isomorphic to $Hom_{\mathfrak{sl}_2}(I, V)$. So we have reduced the above theorem to:

LEMMA 4.3. *For any finite dimensional $\mathfrak{sl}_2$-module V, the functor $V \to V^{\mathfrak{sl}_2}$ is an exact functor.*

The proof of this lemma will rely upon the central Casimir element $C \in U(\mathfrak{sl}_2)$. Note that, by Schur's lemma C will act as a scalar on any finite dimensional irreducible V.

EXERCISE 4.4. If V is irreducible of highest weight m, then C acts as scalar multiplication by $\frac{m^2+2m}{2}$ (hint: it suffices to compute the action of C on a highest-weight vector).

PROPOSITION 4.5. *Let V a finite dimensional $\mathfrak{sl}_2$ module. If C^k acts as 0 on V for some $k > 0$, then $\mathfrak{sl}_2$ acts trivially on V.*

PROOF. We proceed by induction on $\dim V$, the case $\dim V = 0$ being trivial. Let $U \subset V$ be a maximal proper submodule ($U = 0$ is possible). By induction, $\mathfrak{sl}_2 U = 0$. Further, V/U is an irreducible module, and by the above we know that C acts as a nonzero scalar (and hence so does C^k) on V/U unless V/U is the trivial 1 dimensional module. Thus, for $v \in V$, $xv \in U$ for all $x \in \mathfrak{sl}_2$ and so $yxv = 0$ for all $y \in \mathfrak{sl}_2$. Therefore $[x, y]v = 0$; however, since $\mathfrak{sl}_2$ is a simple Lie algebra, we have $[\mathfrak{sl}_2, \mathfrak{sl}_2] = \mathfrak{sl}_2$, and thus V is a trivial module as required. □

REMARK 4.6. [?], [?] The Casimir element C may defined for any finite dimensional semi-simple Lie algebra, using the Killing form. It can be shown that this is a central element which acts nontrivially on nonzero irreducible modules.

Now, the following proposition finishes the argument:

PROPOSITION 4.7. *Let V a finite dimensional $\mathfrak{sl}_2$ module. Then*

(1) *$ker(C) = V^{\mathfrak{sl}_2}$.*
(2) *$ker(C^2) \subseteq ker(C)$.*
(3) *$V = ker(C) \oplus im(C)$.*
(4) *The functor $V \mapsto V^{\mathfrak{sl}_2}$ is exact.*

PROOF. Claim (1) is immediate from Exercise 4.4 above; together with Proposition 4.5, it implies (2). We have $ker(C) \cap im(C) = 0$, by Claim (2), which implies (3). To see (4), we first construct a chain complex $\tilde{V} = 0 \to V \to V \to 0$, where the middle differential is multiplication by C (a morphism because C is central). We have $H_1(\tilde{V}) = H_0(\tilde{V}) \cong V^{\mathfrak{sl}_2}$ by (2). Suppose we have an exact sequence of $\mathfrak{sl}_2$-modules $0 \to U \to V \to W \to 0$. Since $C \in U(\mathfrak{sl}_2)$, the maps necessarily commute with the differentials to give an exact sequence of the complexes:

$$0 \to \tilde{U} \xrightarrow{i} \tilde{V} \xrightarrow{j} \tilde{W} \to 0.$$

We apply the snake lemma to obtain the long exact sequence,

$$0 \to U_0 \xrightarrow{i_1} V^{\mathfrak{sl}_2} \xrightarrow{j_1} W^{\mathfrak{sl}_2} \xrightarrow{\delta} U^{\mathfrak{sl}_2} \xrightarrow{i_0} V^{\mathfrak{sl}_2} \xrightarrow{j_0} W^{\mathfrak{sl}_2} \to 0.$$

Further, the induced map $i_0 : U^{\mathfrak{sl}_2} \to V^{\mathfrak{sl}_2}$ may be identified with the restriction of the original map $U \to W$. By assumption this was injective, and so $im(\delta) = 0$ and the induced right-hand sequence of invariants is exact as required. □

REMARK 4.8. The above proof can be slightly modified to apply to a general semi-simple Lie algebra with Casimir element C.

While Proposition 4.7 guarantees that a general V can be split into a direct sum of simple $\mathfrak{sl}_2$ modules, the following is a more explicit algorithm for constructing the decomposition.

(1) Decompose $V = \oplus V_{(m)}$, where $V_{(m)}$ denotes the eigenspace for the operator C with eignevalue $m^2 + 2m$
(2) Within each $V_{(m)}$, choose a basis $\{v_i\}_{i=1}^k$ for the $\lambda = m$-weight space.
(3) Set $V_{(m),i} = \mathfrak{sl}_2 v_i$, which will be an m dimensional space by our characterization above.
(4) Then $V = \oplus_m(\oplus_i V_{(m),i})$ is a decomposition into simple modules.

5. Characters

The representation theory of $\mathfrak{sl}_2$ admits a powerful theory of characters, analogous to that of finite groups. Computing characters allows us to easily determine the isomorphism type of any finite-dimensional $\mathfrak{sl}_2$-module, and to decompose tensor products.

DEFINITION 5.1. For a finite dimensional $\mathfrak{sl}_2$ module V, we define the formal sum:

$$ch(V) = \sum_{k\in\mathbb{Z}} (\dim V_k) x^k,$$

where we recall that V_k denotes the weight space,

$$V_k = \{v \in V | Hv = kv\}.$$

EXERCISE 5.2. Defining $x^k \cdot x^l = x^{k+l}$, we have:

$$ch(A \oplus B) = ch(A) + ch(B), \quad ch(A \otimes B) = ch(A)ch(B).$$

EXAMPLE 5.3. By Theorem 2.4, we have:

$$ch(V(n)) = \frac{x^{n+1} - x^{-n-1}}{x - x^{-1}} = x^n + x^{n-2} + \cdots + x^{2-n} + x^{-n}.$$

REMARK 5.4. Suppose V is a finite-dimensional $\mathfrak{sl}_2$-module,with character $ch(V)$. Then $p(x) = ch(V)\cdot(x - x^{-1})$ is a Laurent polynomial in x. The coefficient of x^k in $p(x)$ is the multiplicity of the irrreducible $V(k)$ in V.

EXERCISE 5.5. (Clebsch-Gordan) Give a decomposition of $V(m) \otimes V(n)$ as a sum of irreducibles $V(i)$ in two different ways:

(1) by finding all the highest weight vectors in the tensor product.
(2) by computing the character.

EXERCISE 5.6. Show that the subspace $Sym^n(V(1))$ of $V(1)^{\otimes n}$, consisting of symmetric tensors, is a sub-module for the $\mathfrak{sl}_2$ action, and is isomorphic to $V(n)$.

The exercise implies that, as a tensor category, the category of $\mathfrak{sl}_2$-modules is generated by the object $V(1)$: in other words, every irreducible $\mathfrak{sl}_2$-module can be found in some tensor power of $V(1)$.

EXERCISE 5.7. Show that $V(1) \otimes V(1) \cong V(2) \oplus V(0)$.

We will see in next chapter that this is in some sense the only relation in this category.

6. The PBW theorem, and the center of $U(\mathfrak{sl}_2)$

The Poincare-Birkhoff-Witt theorem gives a basis of $U(\mathfrak{g})$ for any Lie algebra $\mathfrak{g}$. The proof we present hinges on a technical result in non-commutative algebra known as the diamond lemma, which is of independent interest.

Let $k\langle X\rangle$ denote the free algebra on a finite set X. Fix a total ordering $<$ on X, extend lexicographically to all monomials of the same degree, and finally declare $m < n$, if m is of lesser degree. Further, fix a finite set S of pairs (m_i, f_i), of a monomial m_i in $k\langle X\rangle$, and a general element $f_i \in k\langle X\rangle$ all of whose monomials are less than m_i, or of smaller degree. A general monomial in $k\langle X\rangle$ is called a PBW monomial if it contains no m_i as a subword. A general element of $k\langle X\rangle$ is called PBW-ordered if it is a sum of PBW monomials.

LEMMA 6.1 (Diamond lemma, [?]). *Suppose that:*

(1) *"Overlap ambiguities are resolvable": For every triple of monomials A, B, C, with some $m_i = AB$, and $m_j = BC$, the expressions f_iC and Af_j can be further resolved to the same PBW-ordered expression.*
(2) *"Inclusion ambiguities are resolvable": For every A, B, C, with $m_i = B$, and $m_j = ABC$, the expressions Af_iC and f_j can be further resolved to the same PBW-ordered expression.*

Then, the set of PBW monomials in $k\langle X\rangle$ forms a basis for the quotient ring $k\langle X\rangle/\langle m_i - f_i | (m_i, f_i) \in S\rangle$.

The defining relations of $U(\mathfrak{sl}_2)$ fit into the above formalism, with $E < H < F$ and:

$$S = \{(FE, EF - H), (HE, EH + 2E), (FH, HF + 2F)\}.$$

THEOREM 6.2 (PBW Theorem). *A basis for $U(\mathfrak{sl}_2)$ is given by the PBW monomials $E^kH^lF^m$, for $k, l, m \in \mathbb{Z}_{\geq 0}$.*

PROOF. We have only to check conditions (1) and (2) from Lemma 6.1. However, (2) is trivially satisfied, since the defining relations are at most quadratic in the generators. In fact, there is only one possible instance of condition (1), which is the monomial FHE. We compute:

$$\begin{aligned}(FH)E &= H(FE) + 2FE = (HE)F - H^2 + 2EF - 2H \\ &= EHF + 2EF - H^2 + 2EF - 2H. \\ F(HE) &= (FE)H + 2(FE) = E(FH) - H^2 + 2EF - 2H \\ &= EHF + 2EF - H^2 + 2EF - 2H.\end{aligned}$$

□

REMARK 6.3. In fact, with only slightly more effort, the diamond lemma and the Jacobi identity together imply a related PBW theorem for any Lie algebra - not necessarily semi-simple - over any field.

COROLLARY 6.4. *We have an isomorphism of $\mathfrak{sl}_2$-modules,*

$$U(\mathfrak{sl}_2) \cong Sym(\mathfrak{sl}_2) := \bigoplus_{k\geq 0} Sym^k(\mathfrak{sl}_2).$$

PROOF. Define a filtration, $\mathcal{F}^\bullet$, of $\mathfrak{sl}_2$-modules on $U(\mathfrak{sl}_2)$ by declaring each of E, H, F to be of degree one. Then it follows from Theorem 6.2 that the associated graded algebra,

$$grU(\mathfrak{sl}_2) = \oplus_{k\geq 0}\mathcal{F}^k U(\mathfrak{sl}_2)/\mathcal{F}^{k-1}U(\mathfrak{sl}_2),$$

is isomorphic to the symmetric algebra, $Sym(\mathfrak{sl}_2)$. However, since each $\mathcal{F}^\bullet$ is a finite-dimensional $\mathfrak{sl}_2$-module, and hence semi-simple, we have an isomorphism $U(\mathfrak{sl}_2) \cong grU(\mathfrak{sl}_2)$. □

COROLLARY 6.5 (Harish-Chandra isomorphism). *The center of $U(\mathfrak{sl}_2)$ is freely generated by the Casimir element. We have an isomorphism:*

$$ZU(\mathfrak{sl}_2) \cong \mathbb{C}[C].$$

PROOF. We present an elementary proof, which highlights the technique of characters. First, it is clear that the powers of C are linearly independent, as the leading order PBW monomial of C^k is E^kF^k. What remains to show is that there are no other central elements. We note that $ZU(\mathfrak{sl}_2)$ may be identified with the space of invariants $U(\mathfrak{sl}_2)^{\mathfrak{sl}_2}$: for $z \in U(\mathfrak{sl}_2)$, we have $[X, z] = 0$ for all X if, and only if, z lies in the center.

Follwoing Corollary 6.4, let us define a weighted character of $U(\mathfrak{sl}_2)$ as follows:

$$\widetilde{ch}(U(\mathfrak{sl}_2)) := \sum_k t^k ch(Sym^k V(2)).$$

As a $\mathbb{C}[H]$-module, we have $V(2) \cong V_{-2} \oplus V_0 \oplus V_2$, which implies an isomorphism of $\mathbb{C}[H]$-modules,

$$Sym(V(2)) \cong Sym(V_{-2}) \otimes Sym(V_0) \otimes Sym(V_2).$$

Thus, we have:

$$\widetilde{ch}(U(\mathfrak{sl}_2)) = \frac{1}{(1-x^{-2}t)(1-t)(1-x^2t)}.$$

The multiplicity of $V(0)$ in each $Sym^kV(2)$ is the xt^k coefficient of $p(x,t) = \widetilde{ch}(U(\mathfrak{sl}_2)) \cdot (x - x^{-1})$, following Remark 5.4. We have:

$$\begin{aligned} p(x,t) &= \frac{x - x^{-1}}{(1-t)(1-x^{-2}t)(1-x^2t)} \\ &= \frac{1}{1-t^2}\left(\frac{x}{1-x^2t} - \frac{x^{-1}}{1-x^{-2}t}\right), \end{aligned}$$

which has x-coefficient $\frac{1}{1-t^2}$. It follows that there are no invariants in odd degrees, and that C^k spans $Sym^{2k}(\mathfrak{sl}_2)$, as desired. □

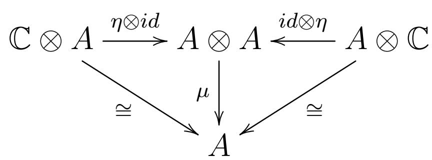
CHAPTER 2

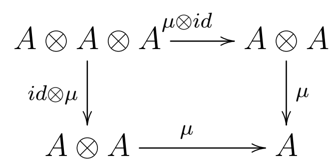
Hopf algebras and tensor categories

1. Hopf algebras

In Example 1.4 of Chapter 1, for any $\mathfrak{g}$-modules V and W, we endowed the vector space $V \otimes W$ with a $\mathfrak{g}$-module structure. In this section, we consider a general class of associative algebras called Hopf algebras, which come equipped with a natural tensor product operation on their categories of modules. The enveloping algebra $U(\mathfrak{sl}_2)$ will be our first example. To begin, let us re-phrase the axioms for an algebra in a convenient categorical fashion.

Definition 1.1. An algebra over $\mathbb{C}$ is a vector space A equipped with a multiplication $\mu : A \otimes A \to A$, and a unit $\eta : \mathbb{C} \to A$, such that the following diagrams commute:





These diagrams represent the unit and associativity axoims, respectively.

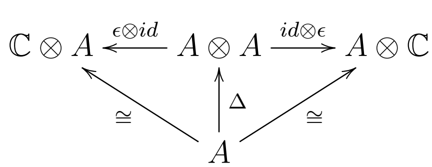
Example 1.2. Given any two algebras A and B, we can define an algebra structure on the vector space $A \otimes B$ by the composition

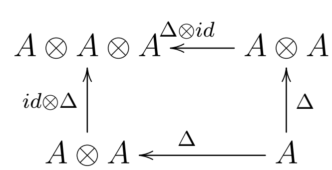
$$A \otimes B \otimes A \otimes B \xrightarrow{id \otimes \tau \otimes id} A \otimes A \otimes B \otimes B \xrightarrow{\mu_A \otimes \mu_B} A \otimes B,$$

where τ flips tensor components: $\tau(v \otimes w) = w \otimes v$.

We define a co-algebra by dualizing the above notions (i.e. by reversing all the arrows).

Definition 1.3. A co-algebra over $\mathbb{C}$ is a vector space A equipped with a co-multiplication $\Delta : A \to A \otimes A$, and a co-unit $\epsilon : A \to \mathbb{C}$, such that the following diagrams commute.





By analogy, these are called the co-unit and co-associativity axioms, respectively.

REMARK 1.4. For any co-algebra A, A^* becomes an algebra, via the composition

$$\mu : A^* \otimes A^* \hookrightarrow (A \otimes A)^* \xrightarrow{\Delta^*} A^*$$

of the natural inclusion , and the dual to the comultiplication map. if A is a finite-dimensional algebra, then A^* becomes a co-algebra, via the composition,

$$\Delta : A \xrightarrow{\mu^*} (A \otimes A)^* \cong A^* \otimes A^*.$$

However, for A infinite dimensional, this prescription does not lead to a comultiplication map for A^*, since the inclusion $A^* \otimes A^* \hookrightarrow (A \otimes A)^*$ is not an isomorphism. In the next chapter we'll see a way around this difficulty.

EXAMPLE 1.5. Given two co-algebras A and B, we can define a co-algebra structure on vector space $A \otimes B$ by

$$A \otimes B \xrightarrow{\Delta \otimes \Delta} A \otimes A \otimes B \otimes B \xrightarrow{id \otimes \tau \otimes id} A \otimes B \otimes A \otimes B$$

DEFINITION 1.6. A *bi-algebra* is a vector space A equipped with algebra structure (A, μ, η) and co-algebra structure (A, Δ, ϵ) satisying either of the conditions:

(1) Δ and ϵ are algebra morphisms.
(2) μ and η are co-algebra morphisms

EXERCISE 1.7. Prove that (1) and (2) are equivalent (hint: write out the appropriate diagrams, and turn your head to one side).

EXERCISE 1.8. Group algebras. Let G be a finite group, and let $\mathbb{C}[G]$ denote its group algebra. Check that $\Delta(g) = g \otimes g$ and $\epsilon(g) = \delta_{e,g}$ defines a bi-algebra structure on $\mathbb{C}[G]$,

EXERCISE 1.9. Enveloping algebra. Let $\mathfrak{g}$ be a Lie algebra, and $U(\mathfrak{g})$ its universal enveloping algebra. For $X \in \mathfrak{g}$, define $\Delta(X) = X \otimes 1 + 1 \otimes X$, and $\epsilon(X) = 0$. Show that this defines a bi-algebra structure on $U(\mathfrak{g})$.

EXERCISE 1.10. Let G be an affine algebraic group, and denote its coordinate algebra $\mathcal{O}(G)$. Define $\Delta(f) \in \mathcal{O}(G) \otimes \mathcal{O}(G) \cong \mathcal{O}(G \times G)$ by $\Delta(f)(x \otimes y) = f(x \cdot y)$, where "$\cdot$" is the multiplication in the group. Define $\epsilon(f)$ as projection onto the constant term. Show that this defines a bi-algebra structure. You will need to show that $\Delta(f)$ is a polynomial in x and y.

Exercise 1.11. Let H be a bialgebra, and let $I \subset H$ be an ideal (with respect to the algebra structure) such that $\Delta(I) \subset H \otimes I + I \otimes H$ (i.e. I is a co-ideal). Show that Δ and ϵ descend, to form a bi-algebra structure on H/I.

Definition 1.12. Let A be a co-algebra, B an algebra. Let $f, g : A \to B$ be linear maps. We define the convolution product $f * g$ as the composition:

$$A \xrightarrow{\Delta} A \otimes A \xrightarrow{f \otimes g} B \otimes B \xrightarrow{\mu} B$$

If A is a bialgebra, then taking $B = A$ above yields the structure of an associative algebra on $\mathrm{End}(A)$, with unit $\eta \circ \epsilon$.

Definition 1.13. A *Hopf algebra* is a bi-algebra H such that there exists an inverse $S : H \to H$ to Id relative to $*$: that is, we have $S * id = id * S = \eta \circ \epsilon$. S is called the *antipode*.

Remark 1.14. Note that the antipode on a bi-algebra is unique, if it exists, by uniqueness of inverses in the associative algebra $\mathrm{End}(A)$.

The best way to understand the antipode is as a sort of linearized inverse, as the following examples illustrate.

Exercise 1.15. Define S for Examples 1.8, 1.9, 1.10, and show that it defines a Hopf algebra in each case.

Exercise 1.16. (**??**, III.3.4) In any Hopf algebra, $S(xy) = S(y)S(x)$. Hint: Define $\nu, \rho \in \mathrm{Hom}(H \otimes H, H)$ by $\nu(x \otimes y) = S(y)S(x)$, and $\rho(x \otimes y) = S(xy)$. Then compute $\rho * \mu = \mu * \nu = \eta \circ \epsilon$.

Remark 1.17. In the case that S is invertible, it is an anti-automorphism and thus can be used to interchange the category of left and right modules over H.

Exercise 1.18. Suppose that the Hopf algebra H is either commutative, or co-commutative. Show by direct computation that $S^2 * S = \eta \circ \tau$, and thus conclude that S is an involution.

Definition 1.19. For any bi-algebra H, and H-modules M and N, we define their tensor product $M \otimes N$ to have as underlying vector spaces the usual tensor product over $\mathbb{C}$, with H-action defined by:

$$H \otimes (M \otimes N) \xrightarrow{\Delta \otimes id} H \otimes H \otimes M \otimes N \xrightarrow{\tau_{23}} H \otimes M \otimes H \otimes N \xrightarrow{\mu_M \otimes \mu_N} M \otimes N$$

Exercise 1.20. Check that $M \otimes N$ is in fact an H-module, by verifying the associativity and unit axioms.

Exercise 1.21. Similarly, given two H-comodules M and N, we can define a comodule structure on their tensor product. Define the action, and check that it gives a well-defined co-module structure.

Remark 1.22. In Examples 1.8, 1.9, 1.10, we recover in this way the usual action on $M \otimes N$. For instance if G is a group, then in $k[G]$, we have $g(v \otimes w) = g(v) \otimes g(w)$; if $\mathfrak{g}$ is a Lie algebra, we have $x(v \otimes w) = x(v) \otimes w + v \otimes x(w)$.

2. The first examples of Hopf Algebras

2.1. The Hopf algebra $U(\mathfrak{sl}_2)$. We have previously defined $U = U(\mathfrak{sl}_2)$ as an algebra; by Example 1.9, we can endow it with a co-product structure such that

$$\Delta(E) = E \otimes 1 + 1 \otimes E,\ \ \Delta(F) = F \otimes 1 + 1 \otimes F$$
$$\Delta(H) = H \otimes 1 + 1 \otimes H,\ \ \epsilon(E) = \epsilon(F) = \epsilon(H) = 0.$$

Following Exercise 1.15, U has antipode given by:

$$S(E) = -E, \quad S(F) = -F, \quad S(H) = -H.$$

2.2. The Hopf algebra $\mathcal{O}(SL_2)$. The algebraic group

$$SL_2 = SL_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid \ a, b, c, d \in \mathbb{C}, ad - bc = 1 \right\}$$

has coordinate algebra $\mathcal{O}(SL_2) := \mathbb{C}[a, b, c, d]/\langle ad - bc - 1\rangle$. We define a co-product for $\mathcal{O} = \mathcal{O}(SL_2)$ on generators as follows:

$$\Delta(a) = a \otimes a + b \otimes c, \quad \Delta(b) = a \otimes b + b \otimes d,$$
$$\Delta(c) = c \otimes a + d \otimes c, \quad \Delta(d) = c \otimes b + d \otimes d.$$

We may write this more concisely as follows:

$$\begin{pmatrix} \Delta(a) & \Delta(b) \\ \Delta(c) & \Delta(d) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Exercise 2.1. Let $\overline{\Delta} : \mathbb{C}[a, b, c, d] \to \mathbb{C}[a, b, c, d] \otimes \mathbb{C}[a, b, c, d]$ be given by the formulas for Δ above. Show that:

(1) $\overline{\Delta}(ad - bc) = (ad - bc) \otimes (ad - bc)$, so that
(2) $\overline{\Delta}(ad - bc - 1) \subset (ad - bc - 1) \otimes H + H \otimes (ad - bc - 1)$.

Conclude that $\overline{\Delta}$ descends to a homomorphism $\Delta : \mathcal{O} \to \mathcal{O} \otimes \mathcal{O}$.

This makes $\mathcal{O}(SL_2)$ into a bi-algebra. We now introduce an antipode, which will endow it with the structure of a Hopf algebra. We define S on generators:

$$\begin{pmatrix} S(a) & S(b) \\ S(c) & S(d) \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

EXERCISE 2.2. Verify that S is an antipode.

3. Tensor Categories

In the previous section, we saw that for any Hopf algebra H, the category of H-modules has a tensor product structure. In this section, we will define the notion of a tensor category, which captures this product structure. The reason for the focus on categorical constructions is that when we look at the quantum analogs of our classical objects, much of the geometric intuition fades, while the categorical notions remain largely intact.

DEFINITION 3.1. Let $\mathcal{C}, \mathcal{D}$ be categories. Their product, $\mathcal{C} \times \mathcal{D}$, is the category whose objects are pairs (V, W), $V \in ob(\mathcal{C}), W \in ob(\mathcal{D})$, and whose morphisms are given by:

$$\mathrm{Mor}((U, V), (U', V')) = \mathrm{Mor}(U, U') \times \mathrm{Mor}(V, V').$$

Let $\otimes$ be a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$. This means that for each pair $(U, V) \in \mathcal{C} \times \mathcal{C}$, we have their tensor product $U \otimes V$, and for any maps $f : U \to U', g : V \to V'$, we have a map $f \otimes g : U \otimes V \to U' \otimes V'$.

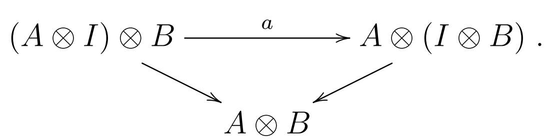
DEFINITION 3.2. An associativity constraint on $\otimes$ is a natural isomorphism $a_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$ which satisfies the *Pentagon Axiom.*

$$\begin{array}{ccc}
((A \otimes B) \otimes C) \otimes D & \xrightarrow[a_{A,B,C} \otimes 1]{} & (A \otimes (B \otimes C)) \otimes D \\
\downarrow{\scriptstyle a_{A \otimes B, C, D}} & & \\
(A \otimes B) \otimes (C \otimes D) & & \downarrow{\scriptstyle a_{A, B \otimes C, D}} \\
\downarrow{\scriptstyle a_{A,B,C \otimes D}} & & \\
A \otimes (B \otimes (C \otimes D)) & \xleftarrow{1 \otimes a_{B,C,D}} & A \otimes ((B \otimes C) \otimes D)
\end{array}$$

REMARK 3.3. It is useful to think of the functor $\otimes$ as a categorified version of an associative product. Whereas in the theory of groups or rings (or more generally, monoids) one encounters the identity $(ab)c = a(bc)$ expressing associativity of multiplication, this is not sensible for categories, as objects are rarely equal, but more often isomorphic (consider the example of tensor products of vector spaces). It is an exercise to show that the basic associative identity for monoids implies that any two parenthesizations of the same word of arbitrary length are equal. In tensor categories, we need to impose an equality of various associators on tensor products of quadruples of objects. MacLane's theorem [] asserts that this commutativity on 4-tuples implies the analogous equality of associators for n-tuples, so that we may omit parenthesizations going forward.

Definition 3.4. A unit for $\otimes$ is a triple (I, l, r), where $I \in \mathcal{C}$, and $l : I \otimes U \to U$ and $r : U \otimes I \to I$ are natural isomorphisms.

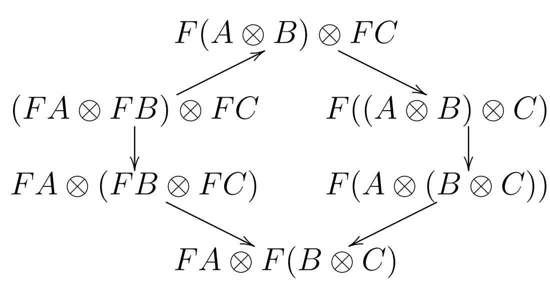
Definition 3.5. A tensor category is a collection $(\mathcal{C}, \otimes, a, I, l, r)$ with a, I, l, r as above, such that we have the following commutative diagram



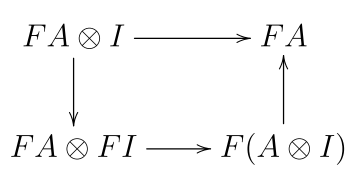
Definition 3.6. A tensor functor $F : (\mathcal{C}, \otimes) \to (\mathcal{D}, \otimes)$ is a pair (F, J) of a functor $F : \mathcal{C} \to \mathcal{D}$, and a natural isomorphism

$$J_{A,B} : FA \otimes FB \xrightarrow{\sim} F(A \otimes B), \qquad I \xrightarrow{\sim} F(I)$$

such that diagrams



and



commute, as well as the similar diagram for right unit constraints.

DEFINITION 3.7. A tensor natural transformation between tensor functors F and G is a natural transformation $\alpha : F \to G$ is such that

$$\begin{array}{ccc} FA \otimes FB & \longrightarrow & F(A \otimes B) \\ \downarrow & & \downarrow \\ GA \otimes GB & \longrightarrow & G(A \otimes B) \end{array}$$

commutes.

DEFINITION 3.8. $(\mathcal{C}, \otimes)$ is *strict* if a, l, r are all equalities in the category (meaning that the underlying objects are equal, and the morphism is the identity). A tensor functor $F = (F, J)$ is strict if J is an equality and $I = FI$.

REMARK 3.9. Most categories arising naturally in representation theory are not strict categories, but we will see in chapter ?? by an extension of MacLane's coherence theorem that any tensor category is tensor equivalent to a strict category. In chapter ??, we will see some examples of strict tensor categories.

EXAMPLE 3.10.

CHAPTER 3

Geometric Representation Theory for SL_2

In this chapter we begin the study of geometric representation theory, in which techniques from algebraic and differential geometry are brought to bear on the representation theory of algebraic groups. We focus on three main results:

(1) the Peter-Weyl Theorem, which states that the coordinate algebra $\mathcal{O}(G)$, viewed as a left $G \times G$-module, contains one direct summand $End(V)$ for every finite dimensional irreducible module V of G;
(2) the Borel-Weil theorem, which realizes finite-dimensional representations of a semi-simple algebraic group geometrically as sections of certain equivariant line bundles on the corresponding flag variety; and
(3) the Beilinson-Bernstein localization theorem, which gives an equivalence between the category of D-modules on the flag variety and the category of $U(\mathfrak{g})$-modules with trivial central character.

As in the previous chapter, we will look to SL_2 for most of our examples.

1. The algebra of matrix coefficients

The finite dimensional representations of a (possibly infinite dimensional) Hopf algebra H determine a natural subalgebra of H^*, called the algebra of matrix coefficients, which is naturally a Hopf algebra, thus overcoming the finiteness issues in Remark **??**. The dual vector space H^* carries an action of $H \otimes H$, given by:

$$((a \otimes b)\phi)(x) := \phi(S(b)xa).$$

Definition 1.1. The external tensor product $V \boxtimes W$ of H-modules V and W is the $H \otimes H$-module with underlying vector space $V \otimes_{\mathbb{C}} W$, and action $(u_1 \otimes u_2)(v \otimes w) := u_1 v \otimes u_2 w$.

Let V be a finite dimensional H-module. For $f \in V^*, v \in V$, the matrix coefficients $c^V_{f,v} \in H^*$ are defubed by $c^V_{f,v}(u) := f(u.v)$, for $u \in H$. The assignment $(f, v) \mapsto c^V_{f,v}$ is bi-linear; we thus obtain a linear map $c^V : V^* \boxtimes V \to H^*$.

Exercise 1.2. Show that $c^V_{f,v} c^W_{g,w} = c^{V \otimes W}_{g \otimes f, v \otimes w}$.

Exercise 1.3. Let $\phi : V \to W$ be a homomorphism of H-modules. Show that, for $v \in V, f \in W^*$, we have $c^W_{f,\phi v} = c^V_{\phi^* f, v}$.

Definition 1.4. The algebra, $\mathcal{O}$, of matrix coefficients, is the linear subspace of H^* spanned by the $c_{f,v}$ for all finite-dimensional. V.

EXERCISE 1.5. Conclude that $\mathcal{O}$ is a $H \otimes H$-submodule of H^*, and that c^V is a $H \otimes H$-module map, by showing, for $a, b \in H$:

$$(a \otimes b)c_{f,v} = c_{bf,av}.$$

EXERCISE 1.6. Fix a basis $v_1, \ldots, v_n$ for V, and let $f_1, \ldots, f_n$ for V^* be the dual basis. Verify that the representation map $\rho : U \to \mathfrak{gl}(V)$ sends x to the matrix $(c_{f_i,v_i}(x))_{i,j=1}^n$, thus justifying the name "matrix coefficient".

EXERCISE 1.7. Suppose that H is commutative, or co-commutative, so that the tensor flip $v \otimes w \mapsto w \otimes v$ is a morphism of H-modules. Show in this case that $\mathcal{O}$ is commutative.

PROPOSITION 1.8. *Let $\Delta : H^* \to (H \otimes H)^*$ denote the dual to the multiplication map on H. Then we have $\Delta(\mathcal{O}) \subset \mathcal{O} \otimes \mathcal{O} \subset (H \otimes H)^*$, and this endows $\mathcal{O}$ with the structure of a Hopf algebra.*

PROOF. For the first claim, it suffices to show that $\Delta c_{f,v} \in \mathcal{O} \otimes \mathcal{O}$, for each finite-dimensional V, each $f \in V^*$, and $v \in V$. Let $\{v_i\}$ be a basis for V and $\{f_i\}$ a dual basis for V^*. The proof follows from the following exercise:

EXERCISE 1.9. Show that $\Delta(c_{f,v}) = \sum_{i=1}^n c_{f,v_i} \otimes c_{f_i,v}$, by checking that this expression satisfies: $\langle \Delta(c_{f,v}), x \otimes y \rangle = \langle c_{f,v}, xy \rangle$.

Having defined the bi-algebra structure, the antipode S is defined by $\langle S(c_{f,v}), x \rangle = \langle c_{f,v}, S(x) \rangle$, for $x \in H$.

□

2. Peter-Weyl Theorem for SL(2)

Returning to the case $U = U(\mathfrak{sl}_2)$, we have the following description of the algebra $\mathcal{O}$ of matrix coefficients.

THEOREM 2.1. *(Peter-Weyl) Let $V(n)$ denote the irreducible representation of $\mathfrak{sl}_2$ of highest weight n. Then we have an isomorphism of $U \otimes U$-modules:*

$$\mathcal{O} \cong \bigoplus_{j=0}^{\infty} V(j)^* \boxtimes V(j),$$

PROOF. We have a map of $U \otimes U$-modules,

$$\bigoplus_{j=0}^{\infty} c^{V(j)} : \bigoplus_{j=0}^{\infty} V(j)^* \boxtimes V(j) \to \mathcal{O}.$$

Each $c^{V(j)}$ is an injection: the kernel is a submodule of the irreducible $U \otimes U$-module $V(j)^* \otimes V(j)$, and each $c^{V(j)}$ is clearly not identically zero. Moreover, the images of $c^{V(j)}$ and $c^{V(k)}$ must intersect trivially, for $j \neq k$, since these are non-isomorphic irreducible submodules.

It only remains to prove surjectivity; we need to show that $\mathcal{O}$ is in fact contained in the sum of of the images of the maps $c^{V(i)}$. For this, let V be an arbitrary finite dimensional representation, and using the semi-simplicity proved in Chapter 1, write V as a finite direct sum of irreducibles:

$$V \cong \bigoplus_{i=0}^{N} V(i)^{\oplus m_i}.$$

Let $\pi_{i,j}$ and $\iota_{i,j}$, respectively, denote the projection onto, and inclusion into, the jth copy of $V(i)$ in the sum. We clearly have $\pi_{i,j}^* = \iota_{i,j}$. Let $f \in V^*, v \in V$. Then we may write:

$$v = \sum_{i,j} \iota_{i,j} v_{i,j}, \qquad f = \sum_{k,l} \pi_{k,l}^* f_{k,l},$$

for some collection of $v_{i,j} \in V(i)$ and $f_{k,l} \in V(i)^*$. Thus, we have:

$$c_{f,v}^V = \sum_{i,j,k,l} c^V_{\pi_{k,l}^* f_{k,l}, \iota_{i,j} v_{i,j}} = \sum_{i,j,k,l} c^V_{f_{k,l}, \pi_{k,l} \iota_{i,j} v_{i,j}}.$$

We have $\pi_{k,l}\iota_{i,j} = \mathrm{Id}_{V(i)}$ if $i = k$, and 0 otherwise. Thus the right hand side lies in the span of the images of the maps $c^{V(i)}$, as desired. □

REMARK 2.2. Clearly, both the statement and proof of the Peter-Weyl theorem apply *mutatis mutandis* for any semi-simple algebraic groups.

3. Reconstructing $\mathcal{O}(SL_2)$ from $U(\mathfrak{sl}_2)$ via matrix coefficients.

Choose a basis v_1, v_2 of $V(1)$, and let v^1, v^2 denote the dual basis of $V(1)^*$. We use the notation $c^i_j := c_{v^i \otimes v_j}$. We denote by i_0 and π_0 the maps:

$$\begin{aligned} i_0 : V(0) &\to V(1) \otimes V(1), & \pi_0 : V(1)^* \otimes V(1)^* &\to V(0) \\ 1 &\mapsto v_1 \otimes v_2 - v_2 \otimes v_1 & \sum a_{ij} v^i \otimes v^j &\mapsto (a_{12} - a_{21}) \end{aligned}$$

Thus i_0 and π_0 are the inclusion and projection, respectively, of the trivial representation relative to the decomposition,

$$V(1) \otimes V(1) \cong V(2) \oplus V(0).$$

EXERCISE 3.1. The purpose of this exercise is to construct an isomorphism between $\mathcal{O}(SL_2)$ and the algebra $\mathcal{O}$ of matrix coefficients on $U(\mathfrak{sl}_2)$.

(1) Show that there exists a unique homomorphism:

$$\phi : \mathbb{C}[a,b,c,d] \to \mathcal{O},$$
$$(a,b,c,d) \mapsto (c_1^1, c_2^1, c_1^2, c_2^2).$$

(2) Show that ϕ is surjective, using the fact that $V(1)$ generates the tensor category of $\mathfrak{sl}_2$ modules.

(3) Show that the relations $c_{f,i_0(v)} = c_{\pi_0(f),v}$, for $f = v^1, v^2$ and $v = v_1, v_2$, reduce to the single relation $ad - bc = 1$.

(4) The algebra $\mathcal{O}(SL_2) = \mathbb{C}[a,b,c,d]/\langle ad - bc - 1\rangle$ admits a filtration with generators a,b,c,d in degree one. Let F_i denote the ith filtration, and show that F_i/F_{i-1} has a basis:

$$\mathcal{B}_i = \{a^k d^l c^m \mid k+l+m = i\} \cup \{a^k d^l b^m \mid k+l+m = i\},$$

so that $\dim F_i/F_{i+1} = |\mathcal{B}_i| = 2\binom{i+2}{2} - (i+1) = (i+1)^2$.

(5) Show that ϕ is a map of filtered vector spaces, where

$$F_i(\mathcal{O}) = \oplus_{k \le i} V(k)^* \boxtimes V(k).$$

(6) Conclude that ϕ is injective, and thus an isomorphism of algebras.

Exercise 3.2. Show that ϕ is a isomorphism of Hopf algebras, by showing that it respects co-products.

Remark 3.3. This exercise is the easiest case of a very general theory, called Tannaka-Krein Reconstruction, which gives a prescription for recovering the coordinate algebra of a reductive algebraic group (more generally, any Hopf algebra) from its category of finite dimensional representations.

4. Equivariant vector bundles, and sheaves

Let X be an algebraic variety over $\mathbb{C}$, and G an algebraic group. Let us denote the multiplication map on G by *mult*:

$$G \times G \xrightarrow{mult} G$$

Suppose G acts on X, meaning that we have an algebraic morphism:

$$G \times X \xrightarrow{act} X$$

which is associative:

$$act \circ (mult \times 1) = (act) \circ (1 \times act) : G \times G \times X \to X$$

Definition 4.1. A G-equivariant vector bundle on X is a vector bundle $\pi : V \to X$, over X, together with an action $G \times V \to V$ commuting with π, and restricting to a linear map $\phi_{g,x} : V_x \to V_{gx}$ of each fiber.

It follows that the maps $\phi_{g,x}$ are linear isomorphisms, and are associative in the following sense:

$$\phi_{h,gx} \circ \phi_{g,x} = \phi_{hg,x}.$$

We will now give a generalization of this definition to sheaves. Using the multiplication, action and projection we can form three maps, $d_0, d_1, d_2 : G \times G \times X \to G \times X$:

$$d_0(g_1, g_2, x) = (g_2, g_1^{-1}x), \quad d_1(g_1, g_2, x) = (g_1 g_2, x),$$

$$d_2(g_1, g_2, x) = (g_1, x).$$

We also have the identity section from $s : X \to G \times X, s(x) = (e, x)$, and the projection $proj : G \times X \to X$, $proj(g, x) = x$.

Definition 4.2. A G-equivariant sheaf on X is a pair $(\mathcal{F}, \theta)$, where $\mathcal{F}$ is a sheaf on X and θ is an isomorphism,

$$\theta : proj^*\mathcal{F} \longrightarrow act^*\mathcal{F}$$

satisfying the cocycle and unit conditions:

$$d_0^*\theta \circ d_2^*\theta = d_1^*\theta, \quad s^*\theta = id_{\mathcal{F}}.$$

Exercise 4.3. Prove that if V is an equivariant vector bundle then the locally free sheaf of sections of V is an equivariant sheaf.

Exercise 4.4. Prove that if V is a G-equivariant locally free sheaf on X, then $\mathrm{Spec}_X(V)$, the associated vector bundle on X is a G-equivariant vector bundle.

Remark 4.5. Note that the we can give this definition also in other categories (topological, differentiable, analytic,...).

Suppose now that $X = Spec(A)$ is an affine variety and $G = Spec(H)$ is an affine algebraic group, so that H is a commutative Hopf algebra. The action of G on X translates into A being a H-comodule algebra:

Definition 4.6. An H-comodule algebra A is an H-comodule, and an algebra, such that the multiplication map $m : A \otimes A \to A$ is a map of comodules, where $A \otimes A$ is an H-module via tensor product.

Definition 4.7. The category $\mathcal{C}_A^H$ of H-equivariant A-modules has as objects H-comodules M, equipped with a map $m : A \otimes M \to M$ of H-comodules, making M into an A-module. The morphisms in this category are the maps that commute with both the A-module structure and the H-comodule structure.

EXERCISE 4.8. In the setup of the preceding paragraph, construct an equivalence between $\mathcal{C}_A^H$ and the category of G-equivariant sheaves on X.

EXERCISE 4.9. Suppose that G acts transitively on X. Show that a G-equivariant sheaf is locally free (hint: produce an isomorphism on stalks, $\mathcal{F}_x \to \mathcal{F}_{gx}$).

EXERCISE 4.10. Let $X = G$, and let G act on itself by left multiplication. Show that the category of quasi-coherent G-equivariant sheaves of $\mathcal{O}_G$-modules is equivalent to the category of vector spaces.

EXERCISE 4.11. Let $X = \{pt\}$ with the trivial G-action. Show that the category of G-equivariant sheaves on X is equivalent to the category of representations of G.

5. Quasi-coherent sheaves on the flag variety

For any semi-simple algebraic group, the flag variety is a homogeneous space, the quotient G/B of G by its Borel subgroup B. In the case $G = SL_2$, the Borel subgroup B is the set of upper-triangular matrices,

$$B = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}.$$

We may identify B with the stabilizer of the line spanned by the first basis vector; the orbit-stabilizer theorem then gives an identification of G/B with the first projective space $\mathbb{P}^1$.

While G/B is a projective variety – in particular, not affine – we can nevertheless approach its category of quasi-coherent sheaves without appeal to projective geometry, by describing quasi-coherent sheaves on G/B as B-equivariant sheaves on G. This purely algebraic point of view will most easily generalize to the quantum case considered in the next chapter, where most of the geometry is necessarily expressed in algebraic terms.

DEFINITION 5.1. The category of quasi-coherent sheaves on the coset space G/B, denoted $\mathcal{QC}oh(G/B)$, has as objects all B-equivariant $\mathcal{O}$-modules on G. Morphisms in $\mathcal{QC}oh(G/B)$ are those which commute with both the $\mathcal{O}$ action and the $\mathcal{O}(B)$-coaction.

REMARK 5.2. It is a theorem due to [] that the flag variety is in fact an algebraic variety, and that furthermore its category of quasi-coherent sheaves is equivalent to the category we have defined above.

REMARK 5.3. Because the G-action is transitive, we can identify the fibers of the sheaf for all $x \in G/B$.

More generally, for any Hopf we have a Hopf algebra H. The next lemma generalizes Exercise 4.10.

Proposition 5.4. *$\mathcal{C}_H^H \sim$ Vect.*

Proof. Let $M^{co-inv} = \{m \in M | \Delta m = m \otimes 1\}$, then $M \mapsto M^{co-inv}$ defines a functor $F : \mathcal{C}_H \to$ Vect. The assignment $V \mapsto V \otimes H$ gives a functor G : Vect $\to \mathcal{C}_H$. To finish the proof, we need to produce natural isomorphisms $M \cong H \otimes M^{co-inv}$ and $(V \otimes H)^{co-inv} \cong V$. □

Suppose H has a quotient Hopf algebra A. We define a category ${}_A\mathcal{C}_H$ as the category whose objects are H-modules M with a right $\mathcal{O}$-comodule and left A-module structures, such that $H \otimes M \to M$ is an A-comodule map and H-comodule map.

Here we use $H \xrightarrow{\Delta} H \otimes H \to A \otimes H$ to give H an A-comodule structure.

Lemma 5.5. *${}_A\mathcal{C}_H \sim$ Left A-modules.*

Proof. This is an easy extension of Proposition 5.4. □

Since $G = SL(2)$ is an *affine* algebraic variety, the quasi-coherent sheaves on G are just the $\mathcal{O}(SL(2))$ modules. In this case, the Borel subgroup is the group U of upper triangular matrices. Thus, we can construct the category of $\mathbb{P}^1$-modules as the category of $\mathcal{O}(SL(2))$ modules M which have $\mathcal{O}(U)$-comodule action, such that $\mathcal{O}(SL(2) \otimes M \to M$ is both an $\mathcal{O}(SL(2))$-module map, and a $\mathcal{O}(U)$-comodule map. This gives us our first description of quasi-coherent modules on $\mathbb{P}^1$.

5.1. The $\mathbb{G}_m$-equivariant construction of $\mathbb{P}^1$. There is a second, less general, construction of quasi-coherent sheaves on $\mathbb{P}^1$, which will give us a more explicit description. We note that $U = T \rtimes N$, where $T \cong \mathbb{C}^\times$ is the group of diagonal matrices, and $N \cong \mathbb{C}$ is the group of unipotent matrices. Thus, $SL(2)/U \cong (SL(2)/N)/T$.

$$N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right\}, U = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right\}, T = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right\}, a, b \in \mathbb{C}.$$

Exercise 5.6. $SL(2)/N \cong \mathbb{A}^2_\circ$, where $\mathbb{A}^2 = Spec(\mathbb{C}[x, y])$, and $\mathbb{A}^2_\circ$ denotes $\mathbb{A}^2 \backslash \{0\}$. It may be helpful to think of $\mathbb{A}^2_\circ$ as the space of based lines $\{(l, v) | 0 \neq v \in l \subset \mathbb{C}^2\}$.

Now let us describe $\mathcal{QC}oh(\mathbb{A}^2_\circ)$. We first recall that since $\mathbb{A}^2$ is affine, $\mathcal{QC}oh(\mathbb{A}^2) = \mathbb{C}[x, y]$-modules.

Definition 5.7. A $\mathbb{C}[x, y]$ module M is *torsion* if for any $m \in M$, there exists an $l >> 0$ s.t. $x^l m = y^l m = 0$.

We consider the restriction functor $Res : \mathcal{QC}oh(\mathbb{A}^2) \to \mathcal{QC}oh(\mathbb{A}^2_\circ)$. This is clearly surjective, since we can always extend a sheaf by zero off of an open set.

LEMMA 5.8. *$Res(M) \cong 0$ if, and only if, M is a torsion sheaf.*

PROOF. Let M be a torsion sheaf on $\mathbb{A}^2$. On $\mathbb{A}^2\backslash\{\text{y-axis}\}$, x is invertible, so M is necessarily zero there. Likewise, on $\mathbb{A}^2\backslash\{\text{x-axis}\}$, y is invertible, so M is zero there. Since these two open sets cover $\mathbb{A}^2_\circ$, we can conclude that torsion sheaves are sent to zero under restriction. Conversely, if M_x and M_y are both zero, then M is a torsion sheaf. □

We would like now to conclude that $\mathcal{QC}oh(\mathbb{A}^2_\circ)$ is the quotient of $\mathcal{QC}oh(\mathbb{A}^2)$ by the full subcategory consisting of torsion modules. In order to say this, we must define what we mean by the quotient of a category by a subcategory. This is naturally defined whenever the categories are abelian, and the subcategory is full, and also closed with respect to short exact sequences. These notions, and the quotient construction, are explained in the appendix **??** on abelian categories.

THEOREM 5.9. *$\mathcal{QC}oh(\mathbb{A}^2_\circ) \simeq \mathbb{C}[x, y] - modules/torsion$.*

THEOREM 5.10. *$\mathcal{QC}oh(\mathbb{P}^1) =$ graded $\mathbb{C}[x, y] - modules/torsion$.*

PROOF. The $\mathbb{C}^*$ action on $\mathbb{C}[x, y]$ is dilation of each homogeneous component, $\lambda(p(x, y)) = \lambda^{deg(p)}p(x, y)$. Thus, an equivariant module with respect to this action inherits a grading $M_k = \{m \in M|\lambda(m) = \lambda^k m\}$. Conversely, given a grading we can define the $\mathbb{C}^*$ action accordingly. □

EXAMPLE 5.11. $\mathbb{C}[x, y]$, which corresponds to $\mathcal{O}_{\mathbb{C}P^1}$;

EXAMPLE 5.12. If $M = \oplus_n M_n$ is an object, then $M(m)$ is defined by the shifted grading, $M(m)_n = M_{n-m}$

EXAMPLE 5.13. The Serre twisting sheaves are a particular case of the last two examples. We have $\mathcal{O}_{\mathbb{C}P^1}(i) = \mathbb{C}[x, y](i)$,

DEFINITION 5.14. We define the global sections functor for a graded $\mathbb{C}[x, y]$-module to just be the zeroeth graded component. $\Gamma(\oplus_n M_n) = M_0$. Clearly, this coincides with the usual definition of global sections of an $\mathcal{O}_{\mathbb{P}^1}$-module.

6. The Borel-Weil Theorem

For an algebraic group G, we say that V is an algebraic module if we have a map to $\mathrm{GL}(V)$ that is a morphism of group varieties. Given an algebraic B-module V, we can obtain another algebraic B-module

$\mathcal{O}(\mathrm{SL}(2)) \otimes V$ by taking the right action of B on $\mathcal{O}(\mathrm{SL}(2))$. This space also has a left $\mathcal{O}(\mathrm{SL}(2))$-module structure. So, we can define an induced $\mathcal{O}(\mathrm{SL}(2))$-module

$$\mathrm{Ind}_B^{\mathrm{SL}(2)}(V) = (\mathcal{O}(\mathrm{SL}(2)) \otimes_{\mathbb{C}} V)^B$$

where the superscript B denotes that we take the B invariant part (only the vectors fixed by B via the action on V and the right action on $\mathcal{O}(\mathrm{SL}(2))$).

We analyze how this induction works in more detail. Since we are considering SL(2), we will only need to work with one-dimensional algebraic B-modules, which we now characterize. A one-dimensional representation of $\mathbb{C}^* \cong \mathbb{G}_m$ is a morphism $\mathbb{C}^* \to \mathbb{C}^*$ respecting multiplication, and it's easy to see that these are the maps $z \mapsto z^n$. There are no non-trivial algebraic representations of $\mathbb{C} \cong \mathbb{G}_a$. Thus, the one-dimensional representations of B are indexed by the integers. We let $\mathbb{C}_n$ denote the representation

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} 1_n = a^{-n}\, 1_n$$

We have the following important result.

Theorem 6.1. *(Borel-Weil)*

$$\mathit{Ind}_B^{SL(2)}\mathbb{C}_n = V(n)^*$$

Proof. Consider the invariants $(\mathcal{O}(\mathrm{SL}(2)) \otimes \mathbb{C}_n)^B$. Note that the B-invariant submodules correspond exactly to irreducible submodules $V(0)$, and hence to highest weight vectors of weight 0. We can use the Peter-Weyl theorem to write

$$(\mathcal{O}(\mathrm{SL}(2)) \otimes \mathbb{C}_n)^B = \left(\bigoplus_{j=0}^{\infty} V(j)^* \otimes V(j) \otimes \mathbb{C}_n \right)^B$$

Note B only acts on the rightmost two factors, so we can reduce to

$$\bigoplus_{j=0}^{\infty} V(j)^* \otimes (V(j) \otimes \mathbb{C}_n)^B$$

Now, for example, if $\{v_0, \ldots, v_j\}$ forms a basis for V_j, then $\{v_0 \otimes 1_n, \ldots, v_j \otimes 1_n\}$ is a basis for $V(j) \otimes \mathbb{C}_n$. The only vector killed by E is $v_o \otimes 1_n$, and it has weight $j - n$. Thus, the only highest weight vectors of weight 0 occur when $j = n$. So, we find $\mathrm{Ind}_B^{\mathrm{SL}(2)}\mathbb{C}_n = V(n)^*$. $\square$

Remark 6.2. More generally the Borel-Weil theorem implies that for G semi-simple, B its Borel sub-algebra, every finite dimensional representation of G can be realized by induction from B in this way.

What is the geometric interpretation of this theorem? We can relate the induced representation to line bundle structures on the quotient $\mathrm{SL}(2)/B$. By proposition **??**, a one dimensional B-module M determines a G-equivariant $\mathcal{O}(G/B)$ line bundle $\tilde{M}$. The global sections $\Gamma(\tilde{M})$ of this line bundle have a G-action, and this module is $Ind_B^G M$. Let's take a look at our example. We can describe quasi-coherent $\mathcal{O}$- modules on $\mathbb{P}^1 \cong \mathrm{SL}(2)/B$ by considering B-equivariant $\mathcal{O}(\mathrm{SL}(2))$-modules. Starting from a B-module V, we can obtain such equivariant modules by tensoring $\mathcal{O}(\mathrm{SL}(2)) \otimes_C V$ and taking the right B-action on $\mathcal{O}(\mathrm{SL}(2))$ as above. For example, starting with $\mathbb{C}_n$, our equivariant module will be $\mathcal{O}(\mathrm{SL}(2)) \otimes \mathbb{C}_n$. By Borel-Weil the global sections of the quotient bundle will be $V(n)^*$, so we can identify this line bundle with the twisting sheaf $\mathcal{O}_{\mathbb{P}^1}(n)$.

7. Beilinson-Bernstein Localization

7.1. D-modules on $\mathbb{P}^1$. In this section, we will construct certain D-modules, which are essentially sets of solutions of algebraic differential equations. In section **??**, we will define D-modules for any affine algebraic variety, but for now, we consider the cases of $\mathbb{A}^2$, $\mathbb{A}^2_\circ = \mathbb{A}^2 \setminus \{0\}$ and $\mathbb{P}^1$. To consider D-modules on a general algebraic variety, one simply sheafifies the construction for affine algebraic varieties.

Definition 7.1. We define the *second Weyl algebra*, W, to be the algebra generated over $\mathbb{C}$ by $\{x, y, \partial_x, \partial_y\}$, subject to relations $[x, \partial_x] = [y, \partial_y] = 1$, with all other pairs of generators commuting. W is a graded algebra over $\mathbb{C}$ with $\deg x = \deg y = 1$, $\deg \partial_x = \deg \partial_y = -1$.

Definition 7.2. A *D-module* on $\mathbb{A}^2$ is a module over W

Definition 7.3. A W-module M is *torsion* if for all $m \in M$, there is a k such that $x^k m = y^k m = 0$

A similar consideration to that which led to quasi-coherent sheaves on $\mathbb{A}^2_\circ$ yields the following

Definition 7.4. *The category of D-modules* on $\mathbb{A}^2_\circ$ is the quotient of the category of W-modules by the full subcategory of torsion modules.

W contains a distinguished element, called the *Euler operator* $T = x\partial_x + y\partial_y$. Geometrically, T corresponds to the vector field on $\mathbb{A}^2$

pointing in the radial direction at every point, and vanishing only at the origin. We now use W to define D-modules on $\mathbb{P}^1$:

DEFINITION 7.5. The category of *D-modules* on $\mathbb{P}^1$ has as its objects graded W_2-modules M modulo torsion such that T acts on the nth graded component M_n as scalar multiplication by n.

REMARK 7.6. This graded action by the Euler operator is the correct notion of equivariance in the differential setting.

EXAMPLE 7.7. The polynomial ring $\mathbb{C}[x, y]$ with the usual grading is a D-module on $\mathbb{P}^1$, where x and y act by left multiplication, and ∂_x and ∂_y act by differntiation. More generally, the structure sheaf is always a D-module.

EXAMPLE 7.8. The shifted modules $\mathbb{C}[x, y](n)$ are *not* D-modules, because although they are modules over W, the Euler operator does not act on the graded components by the correct scalar.

EXAMPLE 7.9. $\mathbb{C}[x, x^{-1}, y]$ with grading $\deg x = \deg y = 1$ and $\deg x^{-1} = -1$ is a D-module. Note that the global sections functor yields $\Gamma(\mathbb{C}[x, x^{-1}, y]) = \mathbb{C}[x^{-1}y]$, whereas above we had $\Gamma(\mathbb{C}[x, y]) = \mathbb{C}$.

7.2. The Localization Theorem. We wish to investigate the structure of W a little further. If we decompose it into graded components as $W = \bigoplus_{i\in\mathbb{Z}} W_i$, then what is the 0th component W_0? Since W acts faithfully on $\mathbb{C}[x, y]$, it suffices to consider the embedding $W \hookrightarrow \text{End}(\mathbb{C}[x, y])$ and answer the same question for the image of W.

EXERCISE 7.10. The component W_0 is generated by the elements $x\partial_y$, $y\partial_x$, $x\partial_x$, and $y\partial_y$.

LEMMA 7.11. *The elements $x^i y^j \partial_x^k \partial_y^l$ form a basis for W_2.*

PROOF. Using the commutation relations, it is easy to show that these elements are stable under left multiplication by the generators of W. Furthermore, since 1 is of this form, these elements must span W. Thus it remains only to check the linear independence of these elements. This is clear from the faithful action on $\mathbb{C}[x, y]$, so we are done. □

Modifying the generating set for W_0 slightly to be $x\partial_y, y\partial_x, T, x\partial_x - y\partial_y$, we now notice a few interesting relations:

$$x\partial_y(x) = 0, \quad y\partial_x(x) = y, \quad (x\partial_x - y\partial_y)(x) = x$$
$$x\partial_y(y) = x, \quad y\partial_x(y) = 0, \quad (x\partial_x - y\partial_y)(y) = -y.$$

This is exactly the action of $\mathfrak{sl}(2,\mathbb{C})$, where we identify the generators $E = x\partial_y$, $F = y\partial_x$, and $H = x\partial_y - y\partial_x$, together with the element T.

DEFINITION 7.12. Let U be a Hopf algebra acting on a module A. Then A is called a *module algebra* if we have a multiplication $\mu : A\otimes A \to A$, which is a map of U-modules. Specifically, if $\Delta u = u_1 \otimes u_2$, then we require $u(ab) = (u_1 a)(u_2 b)$.

For the universal enveloping algebra $U = \mathcal{U}(\mathfrak{sl}(2,\mathbb{C}))$ and $x \in \mathfrak{sl}(2,\mathbb{C})$, we have the comultiplication map $\Delta x = x\otimes 1 + 1\otimes x$, so the definition of a module algebra imposes the condition $x(ab) = (xa)b + a(xb)$. This is precisely the Leibniz rule, so x acts as a derivation. In particular, if $\mathcal{U}(\mathfrak{sl}(2,\mathbb{C}))$ acts on $\mathbb{C}[x,y]$ as a module algebra, then the generators E, F, H act as derivations and so their action coincides with that of $x\partial_y, y\partial_x, x\partial_x - y\partial_y$. (We leave it as an exercise to check that U acts in the correct way.)

In particular, the action of $\mathbb{C}\langle x\partial_y, y\partial_x, x\partial_x - y\partial_y\rangle \subset W \subset \mathrm{End}(\mathbb{C}[x,y])$ is identical to that of $\mathcal{U}(\mathfrak{sl}(2,\mathbb{C}))$. Furthermore, $T = x\partial_x + y\partial_y$ is central inside W_0 since it acts as a scalar on each graded component and thus commutes with these degree-preserving generators there. But we know that the center of $\mathcal{U}(\mathfrak{sl}(2,\mathbb{C}))$ is generated by the Casimir element C, so we can express T as a polynomial in C. Since C acts on $\mathbb{C}[x,y]_i$ as scalar multiplication by $i(i+2)$, and T acts on it as multiplication by i, we must have $C = T^2 + 2T$. Therefore we have

$$W_0 = \mathcal{U}(\mathfrak{sl}(2,\mathbb{C}))[T]/\langle C = T^2 + 2T\rangle.$$

For any D-module M on $\mathbb{P}^1$, we get an action of W_0 on the global sections $\Gamma(M) = M_0$. Since T acts as zero on M_0, however, we see that $\Gamma(M)$ is in fact a module over $\mathcal{U}(\mathfrak{sl}(2,\mathbb{C}))/\langle C = 0\rangle$. This is still an algebra, since C is central and thus $\langle C\rangle$ is a bi-ideal; we will let $\mathcal{U}_0 = \mathcal{U}(\mathfrak{sl}(2,\mathbb{C}))/\langle C = 0\rangle$ for convenience.

EXAMPLE 7.13. If $M = \mathbb{C}[x, x^{-1}, y]$ then $\Gamma(M) = \mathbb{C}(x^{-1}y)$, and clearly C acts on this by 0. We can compute the action of $E, F, H \in \mathfrak{sl}(2,\mathbb{C})$ on this module (as $x\partial_y, y\partial_x, x\partial_x - y\partial_y$ respectively) to see that it is an infinite dimensional $\mathfrak{sl}(2,\mathbb{C})$-module. Taking Fourier transforms gives the dual of the Verma module M_0^*.

We now claim that D-modules over $\mathbb{P}^1$ are equivalent to modules over U_0. More precisely:

PROPOSITION 7.14. *The functor* $\Gamma : D$*-mod*$(\mathbb{P}^1) \to \mathcal{U}_0$*-mod is an equivalence of categories.*

PROOF. Notice that Γ is representable by an object D, i.e. $\Gamma(M) \cong \mathrm{Hom}_{D\text{-mod}}(D, M)$. (We leave it as an exercise to construct this object

$D \in D\text{-mod}(\mathbb{P}^1)$ as a quotient of W_2 by an element T which is defined so that the Casimir element acts the way it should, and to check that $\mathcal{U}_0 = \text{End}(D)$.) Thus we need to prove that D is a projective. We require two facts: first, that $\Gamma = \text{Hom}_{D\text{-mod}}(D, -)$ is exact, and second, that Γ is faithful, or that if $\Gamma(M) = 0$ then $M = 0$.

In order to prove exactness, we first need Kashiwara's theorem: if M is torsion, then $M = \mathbb{C}[\partial_x, \partial_y] \cdot M_0$, where $M_0 = \{m \in M \mid xm = ym = 0\}$. We can check this for modules over $W_1 = \mathbb{C}\langle x, \partial_x\rangle / \langle [\partial_x, x] = 1\rangle$: for any W_1-module M, we define $M_i = \{m \in M \mid x\partial_x m = im\}$. Then we have well-defined maps $x : M_i \to M_{i+1}$ and $\partial_x : M_i \to M_{i-1}$, and $x\partial_x : M_i \to M_i$ is an isomorphism for $i < 0$, so $\partial_x x = x\partial_x + 1$ is an isomorphism on M_i for $i < -1$. But then both $x\partial_x$ and $\partial_x x$ are isomorphisms on M_i, so in particular $x : M_i \to M_{i+1}$ is an isomorphism for $i \leq -2$ and $\partial_x : M_i \to M_{i-1}$ is an isomorphism for $i \leq -1$. In particular, if $xm = 0$, then $x\partial_x m = (\partial_x x - 1)m = -m$ and hence $m \in M_{-1}$. More generally, if $x^i m = 0$ then it follows by an easy induction that $m \in \bigoplus_{j=-i}^{-1} M_j$. We conclude that if M is torsion, then $M = \mathbb{C}[\partial_x] \cdot M_{-1}$, and so the functor $M \mapsto M_{-1}$ gives an equivalence of categories from torsion W_1-modules to vector spaces. An argument by induction will show that the analogous statement is true for any W_i, and so in particular if M is a torsion W_2-module then $M = \mathbb{C}[\partial_x, \partial_y] \cdot M_{-2}$ where $M_{-2} = \{m \in M \mid Tm = -2m\}$. Therefore any graded torsion W_2-module has all homogeneous elements in degrees ≤ -2.

We can now prove that Γ is exact; since it's already left exact, we only need to show that it preserves surjectivity. Suppose that we have an exact sequence $M \to N \to 0$ in the category of graded modules modulo torsion, so that in reality $M \to N$ may not be surjective – all we know is that $C = \text{coker}(M \to N)$ is a graded torsion module. Taking global sections yields a sequence $\Gamma(M) \to \Gamma(N) \to \Gamma(C)$, or $M_0 \to N_0 \to C_0$, and since C is torsion we know that it is concentrated in degrees ≤ -2, so that $C_0 = 0$. But Γ is exact in the graded category, so the sequence $\Gamma(M) \to \Gamma(N) \to 0$ is exact as desired. Therefore Γ is indeed exact. □

EXERCISE 7.15. Complete the proof by showing that Γ is faithful, i.e. that if $M_0 = 0$ then M is torsion.

The representing object D is a $\mathcal{U}_0$-module since $\text{End}(D) = \mathcal{U}_0$, so we now have a *localization functor* $\text{Loc}(M) = D \otimes_{\mathcal{U}_0} M$ on the category of $\mathcal{U}_0$-modules. This passes from an algebraic category to a geometric one, hence in the opposite direction from Γ.

CHAPTER 4

The first quantum example: $U_q(\mathfrak{sl}_2)$.

1. The quantum integers

In this section we introduce some polynomial expressions in a complex variable q, called quantum integers, which share many basic arithmetical properties with the integers. When we define the quantum analogs of SL_2 and $\mathfrak{sl}_2$, the integral weights which arose there will be replaced by quantum integral weights. The study of quantum integers predates quantum physics, and goes back indeed to Gauss, who studied q-series related to finite fields. Only in the last half of the twentieth century have the connections between these polynomials and the mathematics of quantum physics come to be understood. The interested reader should consult [?], [?], [?] for a more thorough exposition.

DEFINITION 1.1. For $a \in \mathbb{Z}$, we define the quantum integer,

$$[a]_q = \frac{q^a - q^{-a}}{q - q^{-1}} = q^a + q^{a-2} + \cdots q^{2-a} + q^{-a} \in \mathbb{C}[q, q^{-1}].$$

We will omit the "q" in the subscript when there is no risk of confusion.

We further define

(1) $[a]! = [a][a-1]\cdots[1]$.
(2) $\begin{bmatrix} a \\ n \end{bmatrix} = \frac{[a]!}{[a-n]![n]!} \in \mathbb{Z}[q]$.

EXERCISE 1.2. Let $(n)_q := q^n[n]_{q^{\frac{1}{2}}} = \frac{q^n - 1}{q-1}$. Let $\mathbb{F}_q$ denote the field with $q = p^k$ elements. Show that:

(1) The general linear group $GL_n(\mathbb{F}_q)$ has order $(n)_q!$.
(2) There are $\binom{n}{k}_q$ subspaces in $\mathbb{F}_q^n$ of dimension k.
(3) Let $D, \bar{D} : \mathbb{C}(q)[x, x^{-1}] \to \mathbb{C}(q)[x, x^{-1}]$ denote the difference operators,

$$(Df)(x) := \frac{f(qx) - f(q^{-1}x)}{x(q - q^{-1})}, \quad (\bar{D}f)(x) := \frac{f(qx) - f(x)}{x(q-1)}.$$

Show that $D(x^n) = [n]x^{n-1}$, and $\bar{D}(x^n) = (n)x^{n-1}$. Observe that $\lim_{q\to 1} D = \lim_{q\to 1} \bar{D} = \frac{d}{dx}$.

2. The quantum enveloping algebra $U_q(\mathfrak{sl}_2)$

DEFINITION 2.1. The quantum enveloping algebra $U_q(\mathfrak{sl}_2)$ is the $\mathbb{C}[q, q^{-1}]$-algebra with generators E, F, K, K^{-1}, with relations:

$$KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F, \quad [E, F] = \frac{K - K^{-1}}{q - q^{-1}}$$

$$KK^{-1} = K^{-1}K = 1.$$

With these relations, we are equipped to prove the quantum analog of the PBW theorem. Declare $E < K < K^{-1} < F$. Then the relations in U_q are of the form:

$$S = \left\{ \begin{array}{c} (K^{\pm 2}E, q^{\pm 1}EK^{\pm 1}), (FK^{\pm 1}, q^{\pm 2}K^{\pm 1}F), (FE, EF - \frac{K-K^{-1}}{q-q^{-1}}), \\ (K^{\pm 1}K^{\mp 1}, 1) \end{array} \right\}.$$

THEOREM 2.2. *(Quantum PBW theorem) The PBW monomials* $\{E^a K^b F^c\}$ *form a basis for* $U_q(\mathfrak{sl}_2)$.

PROOF. It is clear by inspection of the relations that PBW monomials span $U_q(\mathfrak{sl}_2)$. It remains to show that these monomials are linearly independent. Mimicking the proof of the PBW theorem for $U(\mathfrak{sl}_2)$, we need only verify the overlap ambiguities in the statement of the diamond lemma. There is essentially only one interesting relation to check:

$$(FK)E = q^2 KFE = -q^2 \frac{K^2-1}{q-q^{-1}}; \quad F(KE) = q^2 FEK = -q^2 \frac{K^2-1}{q-q^{-1}}.$$

□

COROLLARY 2.3. *U_q has no zero divisors.*

PROOF. This follows by computing the leading order coefficients in the PBW basis. □

REMARK 2.4. Observe that checking the diamond lemma for $U_q(\mathfrak{sl}_2)$ is actually slightly *easier* than for classical $\mathfrak{sl}_2$. We will see that in many ways the relations for $U_q(\mathfrak{sl}_2)$ are easier to work with than for classical $U(\mathfrak{sl}_2)$.

We record the following commutation relations for future use:

LEMMA 2.5. *We have:* $[E, F^m] = \frac{q^{m-1}K - q^{1-m}K^{-1}}{q-q^{-1}}[m]F^{m-1}$.

EXERCISE 2.6. Prove the lemma, using induction and the identity

$$[a, bc] = [a, b]c + b[a, c].$$

An alternative proof of the PBW theorem for quantum $\mathfrak{sl}_2$ may be given by constructing a faithful action of U_q, and verifying linear independence there. To this end, define an action of U_q on the vector space $A = \mathbb{C}[x, y, z, z^{-1}]$ as follows:

$$E(y^s z^n x^r) := y^{s+1} z^n x^r, \quad K(y^s z^n x^r) = q^{2s} y^s z^{n+1} x^r,$$
$$F(y^s z^n x^r) = q^{2n} y^s z^n x^{r+1} + [s] y^{s-1} \frac{zq^{1-s} - z^{-1}q^{s-1}}{q-q^{-1}} z^n x^r.$$

EXERCISE 2.7. Check that this defines an action, and verify that $E^aK^bF^c(1) = y^az^bx^c$. Conclude that the set of PBW monomials is linearly independent.

In what follows, we will assume that $q^n \neq 1$ for all n. The case where q is a root of unity is of considerable interest, and will be addressed in later chapters . Notice that many of the proofs which follow depend on this assumption.

Finally, we note in passing that U_q becomes a graded algebra if we define $\deg(E) = 1, \deg(K) = \deg(K^{-1}) = 0, \deg(F) = -1$.

3. Representation theory for $U_q(\mathfrak{sl}_2)$

The finite-dimensional representation theory for U_q, when q is not a root of unity, is remarkably similar to that of U, as we will see below. Somewhat surprisingly, the representation theory of U_q when q is a root of unity is rather more akin to modular representation theory: this arises from the simple observation that $[m]_q = 0$ if, and only if, $q^{2k} = 1$.

DEFINITION 3.1. A vector $v \in V$ is a weight vector of weight λ if $Kv = \lambda v$. We denote by V_λ the space of weight vectors of weight λ. A weight vector $v \in V_\lambda$ is highest weight if we also have $Ev = 0$.

Observe that $EV_\lambda \subset V_{q^2\lambda}$, $FV_\lambda \subset V_{q^{-2}\lambda}$; hence if q is not a root of unity, and V is finite dimensional, we can always find a highest weight vector.

LEMMA 3.2. *Let $v \in V$ be a h.w.v. of weight λ. Define $v_0 = v$, $v_i = F^{[i]}v_0 = \frac{F^i}{[i]!}v$. Then we have:*

$$Kv_i = q^{-2i}\lambda v_i, \quad Fv_i = [i+1]v_{i+1}, \quad Ev_i = \frac{\lambda q^{-i+1} - \lambda^{-1}q^{i-1}}{q - q^{-1}}v_{i-1}.$$

PROOF. The first two are straightforward computations. For the third, we compute:

$$Ev_i = \frac{EF^i}{[i]!}v_0 = \frac{q^{i-1}K - q^{1-i}K^{-1}}{q - q^{-1}}\frac{F^{i-1}}{[i-1]!}v_0 = \frac{q^{1-i}\lambda - q^{i-1}\lambda^{-1}}{q - q^{-1}}v_{i-1}.$$

□

Now, suppose V is finite dimensional and v_0 is a h.w.v. of weight λ, and $v_m \neq 0, v_{m+1} = 0$. Then, $0 = Ev_{m+1} = [\lambda, -m]v_m$, and thus $[\lambda, -m] = \frac{\lambda q^{-m} - \lambda^{-1}q^m}{q - q^{-1}} = 0$. Hence $\lambda q^{-m} = \lambda q^m$, and $\lambda^2 = q^{2m} \to \lambda = \pm q^m$. In conclusion, we have the following theorem.

THEOREM 3.3. *For each $n \geq 0$, we have two finite dimensional irreducible representations of h.w. $\pm q^n$ of dimension $n+1$, and these are all of the finite dimensional representations.*

4. U_q is a Hopf algebra

In this section we will see that the algebra U_q is equipped with a comultiplication and antipode making it into a Hopf algebra. These will be modelled on the comultiplication and antipodes in $U(\mathfrak{sl}_2)$ from the previous chapter.

PROPOSITION 4.1. *There exists a unique homomorphism of algebras $\triangle : U_q \to U_q \otimes U_q$ defined on generators by*

$$\triangle E = E \otimes 1 + K \otimes E, \quad \triangle F = F \otimes K^{-1} + 1 \otimes F, \quad \triangle K^{\pm 1} = K^{\pm 1} \otimes K^{\pm 1}$$

PROOF. There is no problem defining Δ on the free algebra $T = \mathbb{C}\langle E, F, K, K^{-1}\rangle$. In order for Δ to descend to a homomorphism from $U_q(\mathfrak{sl}_2)$, we need to check $\Delta(J) = 0$ in $U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$. For instance, we must check that:

$$\triangle(EF - FE) = \Delta(\frac{K - K^{-1}}{q - q^{-q}})$$

This we will do now, and leave the remaining relations to the reader to verify.

$$\begin{aligned}
\triangle E \triangle F - \triangle F \triangle E &= (E \otimes 1 + K \otimes E)(F \otimes K^{-1} + 1 \otimes F) \\
&\quad -(F \otimes K^{-1} + 1 \otimes F)(E \otimes 1 + K \otimes E) \\
&= (EF \otimes K^{-1} + E \otimes F + KF \otimes EK^{-1} + K \otimes EF) \\
&\quad -(FE \otimes K^{-1} + E \otimes F + FK \otimes K^{-1}E + K \otimes FE) \\
&= (EF - FE) \otimes K^{-1} + K \otimes (EF - FE) \\
&= \frac{K - K^{-1}}{q - q^{-1}} \otimes K^{-1} + K \otimes \frac{K - K^{-1}}{q - q^{-1}} \\
&= \frac{K \otimes K - K^{-1} \otimes K^{-1}}{q - q^{-1}} \\
&= \triangle(\frac{K - K^{-1}}{q - q^{-1}})
\end{aligned}$$

□

EXERCISE 4.2. Verify that Δ is co-associative, and thus defines a co-multiplication.

We can now define a co-unit ϵ for Δ. Let $\epsilon : U_q \to \mathbb{C}$ be the unique algebra map satisfying $\epsilon(E) = \epsilon(F) = 0, \epsilon(K) = \epsilon(K^{-1}) = 1$.

EXERCISE 4.3. Verify the co-unit axiom for ϵ and Δ.

In conclusion, if we let μ and η be the multiplication and unit maps on the algebra U_q, we have that $(U_q, \mu, \eta, \Delta, \epsilon)$ is a bi-algebra. We have only now to produce an antipode.

PROPOSITION 4.4. *There exists a unique anti-automorphism S of U_q defined on generators by:*

$$S(K) = K^{-1} \quad S(K^{-1}) = K \quad S(E) = -K^{-1}E \quad S(F) = -FK.$$

Furthermore we have $S^2(u) = K^{-1}uK$, for all $u \in U_q$.

PROOF. There is no problem defining S on the free algebra T. To check that S is well defined on U_q then amounts to verifying that $S(J) \subset J$, for which it suffices (since S is an anti-morphism) to check the statement on the multiplicative generators for J. For instance, we must check:

$$S(EF - FE) = S(\frac{K - K^{-1}}{q - q^{-1}}).$$

We do this now, and leave the remaining computations to the reader.

$$\begin{aligned} S(EF - FE) &= S(F)S(E) - S(E)S(F) = FKK^{-1}E - K^{-1}EFK \\ &= FE - EF = \frac{K^{-1} - K}{q - q^{-1}} = S(\frac{K - K^{-1}}{q - q^{-1}}) \end{aligned}$$

The remaining relations follow in similar spirit. □

5. More representation theory for U_q

Now that we have equipped U_q with the structure of a Hopf algebra, its category of representations is endowed with a tensor product, as in (**??**). In the classical case, we saw that the calculus of this tensor product was rather simple, and could be expressed in terms of the Clebsch-Gordan isomorphisms (**??**). In this section we will establish the quantum Clebsch-Gordan isomorphisms, and we will show that the category U_q-mod is semi-simple. The formulations and proofs for both statements will be completely analogous to the classical case.

PROPOSITION 5.1. $V_+(1) \otimes V_+(1) \cong V_+(2) \oplus V_+(0)$

PROOF. We recall the notation of **??**: v_0 denotes a highest weight vector, while $v_1 = Fv_0$. Consider the vector $v = v_0 \otimes v_0 \in V_+(1) \otimes V_+(1)$. We have

$$Ev = Ev_0 \otimes v_0 + Kv_0 \otimes Ev_0 = 0, Kv = Kv_0 \otimes Kv_0 = q^2 v,$$

$$Fv = Fv_0 \otimes K^{-1}v_0 + v_0 \otimes Fv_0 = q^{-1}v_1 \otimes v_0 + v_0 \otimes v_1,$$

$$F^{[2]}v = v_1 \otimes v_1, F^{[3]}v = 0.$$

Finally, we have the vector $w = q^{-1}v_0 \otimes v_1 - v_1 \otimes v_0$, such that $Kw = w$, $Ew = Fw = 0$. □

These two submodules produce the required decomposition.

DEFINITION 5.2. The character $ch_V \in \mathbb{C}[q, q^{-1}]$ of a finite dimensional $U_q(\mathfrak{sl}_2)$-module is the trace of $K|_V$.

EXERCISE 5.3. Verify that $ch_{V(n)} = [n+1]_q$.

EXERCISE 5.4. [?] State and prove the general quantum Clebsch-Gordan formula for $U_q(\mathfrak{sl}_2)$, by mimicking our proof for $U(\mathfrak{sl}_2)$.

EXERCISE 5.5. Let $c_{V(1),V(1)} : V(1) \otimes V(1) \to V(1) \otimes V(1)$ denote the $U_q(\mathfrak{sl}_2)$-linear endomorphism which scales the component $V(2)$ in the tensor product by q, and the component $V(0)$ by $-q^{-1}$. With respect to the basis $v_0 \otimes v_0, v_0 \otimes v_1, v_1 \otimes v_0, v_1 \otimes v_1$ of the tensor product, show that:

$$c_{V(1),V(1)} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q - q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}.$$

5.1. Quantum Casimir element.

DEFINITION 5.6. The quantum Casimir operator, $C \in U_q$ is the element

$$C_q = FE + \frac{Kq + K^{-1}q^{-1}}{(q - q^{-1})^2} = EF + \frac{Kq^{-1} + K^{-1}q}{(q - q^{-1})^2}$$

EXERCISE 5.7. Show that the two definitions of C_q are equal, and that C_q is central.

EXERCISE 5.8. Let $\epsilon \in \{+, -\}$, and let $V_\epsilon(m)$ be a simple U_q module. Then C_q acts by the scalar $\epsilon(\frac{q^{m+1} + q^{-m-1}}{q - q^{-1}})$. In particular, C_q distinguishes between the different $V_\epsilon(m)$

Thus, we have a central element $\tilde{C}_q = C_q - \frac{q + q^{-1}}{(q - q^{-1})^2}$, which acts as zero on a simple module M if and only if it is the trivial module.

THEOREM 5.9. *The category of $U_q(\mathfrak{sl}_2(\mathbb{C}))$-modules is semi-simple.*

PROOF. It is identical to the proof of the classical case **??**, using C_q in place of C. □

Remark 5.10. We have shown that when q is not a root of unity, the category of finite-dimensional type I U_q-modules is equivalent to the category U-mod, as abelian categories. However, as tensor categories, they cannot be equivalent, because the co-product is non-cocommutative in U_q.

Remark 5.11. For any M a finite dimensional U_q-module, we can decompose $M = M_+ \oplus M_-$, where M_+ is a sum of type I modules, M_- is a sum of type II modules. Finally, we observe in passing that $V_-(m) \cong V_-(0) \otimes V_+(m)$.

6. The locally finite part and the center of $U_q(\mathfrak{sl}_2)$

There is a peculiarity in the construction of $U_q(\mathfrak{sl}_2)$. As with any Hopf algebra, we may consider the "adjoint" action of $U_q(\mathfrak{sl}_2)$ on itself:

$$x \cdot y := x_{(1)} y S(x_{(2)}),$$

where $\Delta(x) = x_{(1)} \otimes x_{(2)}$ (the implicit sum is suppressed in the notation). In the classical setting, the adjoint action is just the commutator action, and we found (via the PBW theorem) that $U(\mathfrak{sl}_2)$ decomposed naturally as a direct sum of finite dimensional representations. In particular, for any given $x \in U(\mathfrak{sl}_2)$, the orbit $U(\mathfrak{sl}_2) \cdot x$ of x was finite-dimensional. For a Hopf algebra H, we let H' denote the sub-space of elements x which generate a finite orbit under the adjoint action.

For $U_q(\mathfrak{sl}_2)$, we compute:

$$\begin{aligned} &E \cdot (E^l K^m F^n) \\ &\quad = E^{l+1} K^m F^n - K E^l K^m F^n K^{-1} E \\ &\quad = (1 - q^{2l-2n+2m}) E^{l+1} K^m F^n + q^{2l-2n} E^l K^m \frac{q^{n-1}K - q^{1-n}K^{-1}}{q - q^{-1}} [n] F^{n-1}. \\ &F \cdot (E^l K^m F^n) \\ &\quad = F E^l K^m F^n K - E^l K^m F^n F K \\ &\quad = q^{2n}(q^{-2m} - q^2) E^l K^{m+1} F^{n+1} - q^{2n} [l] E^{l-1} \frac{K q^{n-1} - K^{-1} q^{1-n}}{q - q^{-1}} K^{m+1} F^n. \end{aligned}$$

It follows easily that the locally finite part $U_q'(\mathfrak{sl}_2)$ of $U_q(\mathfrak{sl}_2)$ is generated by the elements EK^{-1}, F, K^{-1}. Let us define:

$$\bar{E} = EK^{-1}, \quad \bar{F} = F, \quad K^{-1}, \quad \bar{L} = \frac{1 - K^{-2}}{q - q^{-1}}.$$

We can easily compute commutation relations amonst generators of $U'(\mathfrak{sl}_2)$:

$$\bar{E}\bar{F} - \bar{F}\bar{E} = \frac{1 - K^{-2}}{q - q^{-1}} = \bar{L}. \tag{3}$$

$$q^4\bar{L}\bar{E} - \bar{E}\bar{L} = \frac{q^4\bar{E} - q^4K^{-2}\bar{E} - \bar{E} + \bar{E}K^{-2}}{q - q^{-1}} = q^2[2]\bar{E}. \tag{4}$$

$$\bar{L}\bar{F} - q^4\bar{F}\bar{L} = \frac{F - K^{-2}F - q^4F - q^4FK^{-2}}{q - q^{-1}} = -q^2[2]\bar{F}. \tag{5}$$

$$(q - q^{-1})\bar{L} = 1 - K^{-2}. \tag{6}$$

PROPOSITION 6.1. *The algebra $U'_q(\mathfrak{sl}_2)$ is freely generated by $\bar{E}$, $\bar{F}$, $\bar{L}$, and K^{-1}, subject to relations* (3)-(6).

PROPOSITION 6.2. *The specialization $U'_1(\mathfrak{sl}_2)$, of $U'_q(\mathfrak{sl}_2)$ at $q = 1$, is isomorphic to $U(\mathfrak{sl}_2) \otimes \mathbb{C}[\mathbb{Z}/2]$, via:*

$$\phi : U'_1(\mathfrak{sl}_2) \to U(\mathfrak{sl}_2) \otimes \mathbb{C}[\mathbb{Z}/2],$$
$$(\bar{E}, \bar{F}, \bar{L}, K^{-1}) \mapsto (E, F, H, \epsilon),$$

where ϵ is the non-trivial element in $\mathbb{Z}/2$.

PROPOSITION 6.3. *We have an isomorphism,*

$$U'_q(\mathfrak{sl}_2) \cong \mathbb{C}[\mathbb{Z}/2] \otimes \left(\bigoplus_{k \geq 0} Sym_q^k V(1) \right).$$

COROLLARY 6.4. *The center of $U_q(\mathfrak{sl}_2)$ is the subalgebra freely generated by C_q.*

PROOF. The center of any Hopf algebra coincides with the ad-invariant part, and so is clearly contained in U'_q. The character computation of Chapter 1 therefore applies mutatis mutandis. $\square$

CHAPTER 5

Categorical Commutativity in Braided Tensor Categories

1. Braided and Symmetric Tensor Categories

The Hopf algebras appearing in classical representation theory are either commutative as an algebra, or co-commutative as a co-algebra. Their quantum analogs are clearly no longer commutative, nor co-commutative; however they satisfy a weaker condition called "quasi-triangularity", which we now explore.

Let H be a Hopf algebra, and consider the tensor product $V \otimes W$ of H-modules V and W. We have the map $\tau : V \otimes W \to W \otimes V$ of vector spaces, which simply switches the tensor factors, $\tau(v \otimes w) = w \otimes v$.

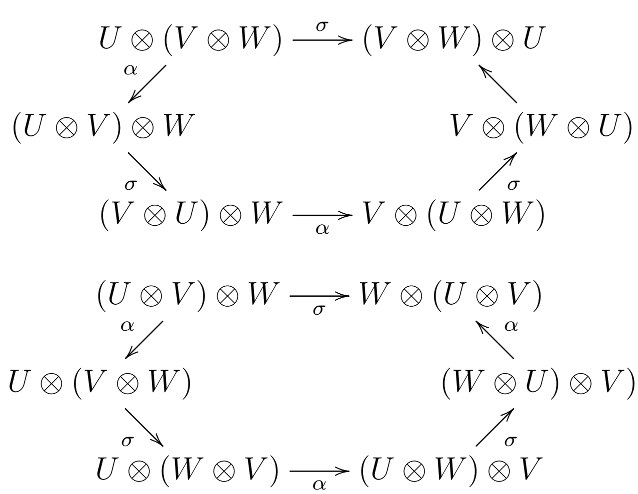
Exercise 1.1. Show that τ is a morphism of H-modules for all $V, W \in H$-mod if, and only if, H is either commutative or co-commutative. (hint: consider the left regular action of H on itself)

The tensor flip τ is not a map of $U_q(\mathfrak{sl}_2)$-modules, as $U_q(\mathfrak{sl}_2)$ is neither commutative, nor co-commutative. Nevertheless, in this chapter, we construct natural isomorphisms $\sigma_{V,W} : V \otimes W \to W \otimes V$ generalizing $\sigma_{V(1),V(1)}$ from Chapter 4, and satisfying a rich set of axioms endowing the category $\mathcal{C} = U_q(\mathfrak{sl}_2) - mod_f$ of finite dimensional $U_q(\mathfrak{sl}_2)$-modules with the structure of a *braided tensor category*.

Definition 1.2. Let $(\mathcal{C}, \otimes, a, l, r)$ be a tensor category. A commutativity constraint σ on $\mathcal{C}$ is a natural isomorphism,

$$\sigma_{V,W} : V \otimes W \to W \otimes V,$$

for $V, W \in \mathcal{C}$, such that for all U, V, W the following diagrams commute.



A braided tensor category is a tensor category, together with a commutativity constraint σ.

REMARK 1.3. Like the pentagon axiom for the associator, the hexagon axiom resolves a potential ambiguity. To move W past $U \otimes V$, one can either group U and V, and commute W past them jointly, or pass them one at a time. The hexagon diagram axiom asserts that in either manner, you obtain the same isomorphism.

REMARK 1.4. More concisely, a commutativity constraint is the data necessary to equip the identity functor, $\mathrm{Id} : \mathcal{C} \to \mathcal{C}^{op}$, from $\mathcal{C}$ to the category $\mathcal{C}^{op}$ with the structure of a tensor functor, where $\mathcal{C}^{op}$ denotes the same underlying abelian category as $\mathcal{C}$, but with $V \otimes^{op} W := W \otimes V$.

PROPOSITION 1.5. *Let $\mathcal{C}$ be a braided tensor category, and let $U, V, W \in \mathcal{C}$. Then (suppressing associators), we have the following equality in $Hom_{\mathcal{C}}(U \otimes V \otimes W, W \otimes V \otimes U)$:*

$$\sigma_{V,W} \circ \sigma_{U,W} \circ \sigma_{U,V} = \sigma_{U,V} \sigma_{U,W} \sigma_{V,W}.$$

PROOF. The naturality of σ in each argument implies:

$$\sigma_{V \otimes U,W} \circ \sigma_{V,W} \otimes \mathrm{Id}_W = \sigma_{V,W} \circ \sigma_{U \otimes V,W}.$$

Applying the hexagon axiom to each of $\sigma_{V \otimes U,W}$ and $\sigma_{U \otimes V,W}$, we obtain asserted equality. □

DEFINITION 1.6. A braided tensor category $\mathcal{C}$ is *symmetric* if for each $V, W \in \mathcal{C}$, we have $\sigma_{V,W} \circ \sigma_{W,V} = \mathrm{Id}$.

EXERCISE 1.7. Let H be a commutative or co-commutative Hopf algebra. Check that $\sigma = \tau$ is a commutativity constraint on H-mod, and that it squares to the identity, so that H-mod is a symmetric tensor category.

EXERCISE 1.8. Denote by S_n the symmetric group on n letters, generated by adjacent swaps $s_{i,i+1}$. Let $V \in \mathcal{C}$ be an element of a symmetric tensor category. Show that the map $s_{i,i+1} \mapsto Id. \otimes \cdots \otimes \sigma_{V,V} \otimes \cdots \otimes Id.$ defines a homomorphism of S_n to $End(V^{\otimes n})$. In the case $\mathcal{C} = H$-mod, and $\dim_{\mathbb{C}} V \geq n$, show that this is an inclusion. (Hint: consider a basis $e_1, \ldots, e_n$, and argue that the stabilizer of $e_1 \otimes \cdots \otimes e_n$ is trivial).

When q is not a root of unity, we exhibited in Chapter ?? an equivalence of abelian categories between the category of finite dimensional type-I $U_q(\mathfrak{sl}_2)$-modules and that of the finite dimensional $U(\mathfrak{sl}_2)$-modules. There we observed that as tensor categories these two are not equivalent, because $U_q(\mathfrak{sl}_2)$ is non-cocommutative.

2. R-matrix Preliminaries

In this section we answer the natural question: what is the necessary structure on a Hopf algebra H, to endow $\mathcal{C} = H$-mod with the structure of a braided tensor category? To answer this question, let us suppose that the category H-mod is braided, and consider the left regular actoin of H on itself. We have a braiding

$$\sigma_{H,H} : H \otimes H \to H \otimes H.$$

We define $R := \tau\sigma_{H\otimes H}(1 \otimes 1)$. Given arbitrary H-modules M and N, and arbitrary elements $m \in M$,$n \in N$, we have a homomorphism of H-modules,

$$\mu_{m,n} : H \otimes H \to M \otimes N,$$

$$h_1 \otimes h_2 \mapsto h_1 m \otimes h_2 n.$$

By naturality of σ, we must have $\sigma_{M,N}(m \otimes n) = \tau R(m \otimes n)$.

Remark 2.1. For historical reasons relating to their physics origins, braiding operators are often called R-matrices. Elements $R \in H \otimes H$ obtained in this way are called "universal R-matrices", as their action on any $V \otimes W$ is an R-matrix.

Exercise 2.2. Show that the element R is invertible and satisfies $\Delta^{op}(u) = R\Delta(u)R^{-1}$, where $\Delta^{op} = \tau_{H,H} \circ \Delta$, or in Sweedler's notation, $\Delta^{op}(u) = u_{(2)} \otimes u_{(1)}$. Hint: Apply the H-linearity of $c_{V,W}$

Exercise 2.3. Show that the hexagon relations imply the identity $(\Delta \otimes id)(R) = R_{13}R_{23}$ and $(id \otimes \Delta)(R) = R_{13}R_{12}$, where for $R = \sum s_i \otimes t_i$, we define:

$$R_{13} := \sum s_i \otimes 1 \otimes t_i, \quad R_{23} := \sum 1 \otimes s_i \otimes t_i, \quad R_{12} = \sum s_i \otimes t_i \otimes 1.$$

Definition 2.4. A quasi-triangular Hopf algebra is a Hopf algebra H, equipped with an invertible element $R \in H \otimes H$, such that $\Delta^{op}(u) = R\Delta(u)R^{-1}$ for all $u \in H$, and satisfying $(\Delta \otimes id)(R) = R_{13}R_{23}$ and $(id \otimes \Delta)(R) = R_{13}R_{12}$.

Exercise 2.5. Let H be a quasi-triangular Hopf algebra, with H-modules M and N. Define H-module homomorphisms,

$$\sigma_{M,N} : M \otimes N \to N \otimes M,$$

$$\sigma(m \otimes n) := \tau(R(m \otimes n)).$$

Prove that σ defines a braiding on the category of H-modules.

We have shown that the data of a braiding on the category of H-modules is equivalent to that of a quasi-triangular structure on H.

3. Drinfeld's Universal R-matrix

The first universal R-matrix for $U_q(\mathfrak{sl}_2)$ was given by Drinfeld [?]. Drinfeld's solution expresses the universal R-matrix for U_q not as living in $U_q \otimes U_q$, but rather in a $\hbar$-adic completion of $U_q[[H,h]]\widehat{\otimes} U_q[[H,h]]$, where H, h are formal parameters satisfying $q = e^{\hbar/2}$, and $K = \exp\frac{\hbar H}{2}$:

$$R = (\sum_{n=0}^{\infty} \frac{(1-q^2)^n}{[n]!} q^{-\frac{n(n-1)}{2}} E^n \otimes F^n) \exp(\frac{\hbar}{4} H \otimes H)$$

Drinfeld's construction of the R-matrix is perhaps best understood by regarding $U_q(\mathfrak{sl}_2)$ as a certain quotient of the Drinfeld double $D(U_q(\underline{)}$ of its Borel sub-algebra. Discussion of $\hbar$-adic completion, and the Drinfeld double construction, would take us too far afield. We refer the interested reader instead to Kassel.

4. Lusztig's R-matrices

Lusztig's approach to defining the universal R-matrix, like Drinfeld's, involves an infinite sum, but one which evaluates to a finte sum on $V \otimes W$, whenever either V or W is a finite dimensional $U_q(\mathfrak{sl}_2)$-module. It will be clear from the construction that Lusztig's and Drinfeld's constructions agree, upon substituting $q = e^{\frac{\hbar}{2}}$, and $K = e^{\hbar H}$.

To begin, we define elements Θ_n in $U_q \otimes U_q$:

$$\Theta_n = a_n E^n \otimes F^n, \quad a_n = (-1)^n q^{-\frac{n(n-1)}{2}} \frac{(q-q^{-1})^n}{[n]!}$$

For example, $\Theta_0 = 1 \otimes 1$, $\Theta_1 = -(q-q^{-1})E \otimes F$. And we have

$$a_n = -q^{-(n-1)} \frac{q-q^{-1}}{[n]} a_{n-1}.$$

EXERCISE 4.1. Prove the following identities:

$$(1 \otimes E)\Theta_n + (E \otimes K)\Theta_{n-1} = \Theta_n(1 \otimes E) + \Theta_{n-1}(E \otimes K^{-1})$$
$$(F \otimes 1)\Theta_n + (K^{-1} \otimes F)\Theta_{n-1} = \Theta_n(F \otimes 1) + \Theta_{n-1}(K \otimes F)$$
$$(K \otimes K)\Theta_n = \Theta_n(K \otimes K)$$

EXERCISE 4.2. Let α be an algebra anti-automorphism of a Hopf algebra H, and define

$$\Delta^\alpha = \tau(\alpha \otimes \alpha) \circ \Delta \circ \alpha^{-1}, \quad \varepsilon^\alpha = \varepsilon \circ \alpha^{-1}, \quad S^\alpha = \alpha \circ S \circ \alpha^{-1}.$$

Show that these define a Hopf algebra structure on H.

EXERCISE 4.3. There exists a unique antiautomorphism $\alpha : U_q \to U_q$ such that $\alpha(E) = E, \alpha(F) = F, \alpha(K) = K^{-1}$.

Thus we can use this antiautomorphism to define an alternate Hopf algebra structure on U_q,

$$\Delta^\alpha(E) = 1\otimes E + E\otimes K^{-1},\ \Delta^\alpha(F) = K\otimes F + F\otimes 1,\ \Delta^\alpha(K) = K\otimes K.$$

DEFINITION 4.4. Define the linear operator $\Theta : M \otimes N \to M \otimes N$ by

$$\Theta = \sum_{n\geq 0} \Theta_n.$$

Because E, F act locally nilpotently on any locally finite module, this infinte sum is in fact a finite sum when applied to any vector, and thus is well-defined in $End_{\mathbb{C}}(M \otimes N)$. Because $\Theta = 1 \otimes 1 +$ (locally nilpotent operators) is unipotent, we have that Θ is invertible.

REMARK 4.5. For any $u \in U_q$, we have an equality of the linear maps

$$\Delta^{op}(u)\Theta = \Theta\Delta^\alpha(u).$$

If the righthand side were $\Delta(u)$, instead of $\Delta^\alpha(u)$, then Θ would satisfy the same relations as a universal R-matrix **??**. This modification is accomplished in Drinfeld's construction by the multiplying by the infinite series $\exp(\frac{h}{4}H\otimes H)$. However, as we will see, Lusztig's solution still gives an R-matrix when restricted to the locally finite U_q-modules.

EXERCISE 4.6. We compute the matrix Θ explicitly for the module $V(1) \otimes V(1)$. Choose the standard basis for $V(1) = \text{span}\{v_0, v_1\}$ and $V(1) \otimes V(1) = \text{span}\{v_0 \otimes v_0, v_0 \otimes v_1, v_1 \otimes v_0, v_1 \otimes v_1\}$. Deduce:

$$\Theta_0 + \Theta_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & q^{-1} - q & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

5. Weights of Type I, and bicharacters (needs better title)

DEFINITION 5.1. A finite dimensional module for U_q is type I if all weight spaces are in $\Lambda = \{q^n, n \in \mathbb{Z}\}$.

DEFINITION 5.2. We denote by $\chi(M)$ the character of M, which is the formal sum $\chi(M) = \sum dim M_{q^i} z^i$. We note that the $\chi(V(n))$'s are linearly independent and $M \cong N$ if, and only if $\chi(M) = \chi(N)$.

DEFINITION 5.3. A *bi-character* is a map $f : \Lambda \times \Lambda \to \mathbb{C}^\times$ s.t.

$$\begin{aligned} f(\lambda\lambda', \mu) &= f(\lambda, \mu) f(\lambda', \mu), \\ f(\lambda, \mu\mu') &= f(\lambda, \mu) f(\lambda, \mu'), \\ f(\lambda, \mu) &= \lambda f(\lambda, \mu q^2) = \mu f(\lambda q^2, \mu). \end{aligned}$$

Then we have

$$f(q^a, q^b) = f(q,q)^{ab}, \ f(q,q)f(q,q) = f(q,q^2) = q^{-1}f(q,1) = q^{-1},$$

thus $f(q,q)$ is a square root of q^{-1}.

EXERCISE 5.4. Choose a square root of q, and define $f(q^a, q^b) = q^{-\frac{ab}{2}}$. Check that this gives a bi-character

For any finite dimensional U_q-modules M, N, define $\tilde{f} : M \otimes N \to M \otimes N$ as follows:

$$\text{for } m \in M_\lambda, \ n \in N_\mu, \ \tilde{f}(m \otimes n) = f(\lambda, \mu)(m \otimes n).$$

LEMMA 5.5. *Let* $\Theta^f = \Theta \circ \tilde{f}$*, then we have* $\Delta^{op}(u) \circ \Theta^f = \Theta^f \circ \Delta(u)$.

PROOF. We need to check that $f \circ \Delta(u) = \Delta^\alpha(u) \circ f$, which we may verify on the generators E, K, F. We give the proof for E; the proof for F is similar, and the proof for K is trivial. We compute:

$$\begin{aligned} f \circ \Delta(u)(m \otimes n) &= f(q^2\lambda, \mu)Em \otimes n + f(\lambda, q^2\mu)\lambda m \otimes En \\ &= f(\lambda,\mu)(\mu^{-1}Em \otimes n + m \otimes En) \\ &= \Delta^\alpha(E) \circ f(m \otimes n). \end{aligned}$$

□

As a consequence, we have:

THEOREM 5.6. *The map* $\sigma_{M,N} = \tau \circ \Theta^f : M \otimes N \to N \otimes M$ *is an isomorphism of* U_q*-modules.*

THEOREM 5.7. *The isomorphisms* $\sigma = \tau \circ \Theta^f$ *satisfy the hexagon relations*??.

DEFINITION 5.8. Let $\Theta_n^{'} = a_n K^n \otimes E^n \otimes F^n$ and $\Theta_n^{''} = a_n E^n \otimes F^n \otimes K^{-n}$.

CLAIM 5.9. *We have:*

$$(\Delta \otimes 1)(\Theta_n) = \sum_{i=0}^{n} (\Theta_{n-i})_{13}\Theta_i^{'}, \quad (1 \otimes \Delta)(\Theta_n) = \sum_{i=0}^{n} (\Theta_{n-i})_{13}\Theta_i^{''}.$$

PROOF. We begin by computing the coproduct on powers of E and F. We have:

$$\begin{aligned} \Delta(E^n) &= \sum_{i=0}^{r} q^{i(r-i)} \begin{bmatrix} r \\ i \end{bmatrix} E^{r-i}K^i \otimes E^i \\ \Delta(F^n) &= \sum_{i=0}^{r} q^{i(r-i)} \begin{bmatrix} r \\ i \end{bmatrix} F^i \otimes F^{r-i}K^{-i}. \end{aligned}$$

The proof is an instance of the q-binomial theorem, for the q-commuting pairs $(E \otimes 1,\ K \otimes E)$ and $(F \otimes K^{-1},\ 1 \otimes F)$. Now, we know that

$$(1 \otimes \Delta)(\Theta_n) = a_n(E^n \otimes \Delta(F^n)).$$

Applying the above formula then yields:

$$(1 \otimes \Delta)(\Theta_n) = \sum_{j=0}^{n} q^{-j(n-j)} \begin{bmatrix} n \\ j \end{bmatrix} a_n E^n \otimes F^j \otimes K^{-j} F^{n-j}$$

On the other hand, we compute:

$$\begin{aligned}\sum_{i=0}^{n} (1 \otimes \Theta_{n-i})\Theta_i^{''} &= \sum_{j=0}^{n} a_{n-j} a_j E^n \otimes F^j \otimes F^{n-j} K^{-j}) \\ &= \sum_{j=0}^{n} a_{n-j} a_j q^{-2j(n-j)} (E^n \otimes F^j \otimes K^{-j} F^{n-j}).\end{aligned}$$

The claimed identity now follows from the identity:

$$q^{-2j(n-j)} a_j a_{n-j} = q^{-j(n-j)} \begin{bmatrix} n \\ j \end{bmatrix} a_n,$$

which is an easy computation from the definitions. The second formula follows from similar computations. □

Now, the proof of the main theorem uses formulas obtained from the above via twisting by α. We have

$$(\alpha \otimes \alpha)(\Theta_n) = \Theta_n, \quad \tau_{12,3}(\alpha \otimes \alpha \otimes \alpha)(\Theta_n^{'}) = \Theta_n^{''}.$$

Applying $(\alpha \otimes \alpha \otimes \alpha)$ to the above equations thus yields

$$\begin{aligned}(\Delta^{\alpha} \otimes 1)(\Theta_n) &= \sum_{i=0}^{n} \Theta_i^{'} (1 \otimes \Theta_{n-i}) \\ (1 \otimes \Delta^{\alpha})(\Theta_n) &= \sum_{i=0}^{n} \Theta_i^{''} (\Theta_{n-i} \otimes 1)\end{aligned}$$

We shall also need several more identities: if we define $\tilde{f}_{1,2}$ to be $\tilde{f} \otimes 1$ (and similarly for $\tilde{f}_{2,3}$ and $\tilde{f}_{1,3}$), then we have the relations $\tilde{f}_{1,2}\Theta_{1,3} = \Theta^{'}\tilde{f}_{1,2}$ and $\tilde{f}_{2,3}\Theta_{1,3} = \Theta^{''}\tilde{f}_{2,3}$, where $\Theta^{'} = \sum_n \Theta_n^{'}$ and similarly for $\Theta^{''}$; these relations follow immediately from the multiplicative properties of $\tilde{f}$. Further, one also easily derives

$$\begin{aligned}\tilde{f}_{1,2}\tilde{f}_{2,3}(1 \otimes \Theta) &= (1 \otimes \Theta)\tilde{f}_{1,2}\tilde{f}_{2,3} \\ \tilde{f}_{2,3}\tilde{f}_{1,3}(\Theta \otimes 1) &= (\Theta \otimes 1)\tilde{f}_{2,3}\tilde{f}_{1,3}\end{aligned}$$

To conclude that the Yang-Baxter equation holds, we write out both sides; the left hand being

$$(\Theta \otimes 1)\tilde{f}_{1,2}\Theta_{1,3}\tilde{f}_{1,3}(1 \otimes \Theta)\tilde{f}_{2,3}$$

and the right hand being

$$(1 \otimes \Theta)\tilde{f}_{2,3}\Theta_{1,3}\tilde{f}_{1,3}(\Theta \otimes 1)\tilde{f}_{1,2}$$

Now, using the above relations to rearrange the left hand side, one gets

$$(\Theta \otimes 1)(\Theta^{'})(1 \otimes \Theta)\tilde{f}_{1,3}\tilde{f}_{1,2}\tilde{f}_{2,3}$$

To deal with this, we rearrange the Θ terms as follows:

$$\begin{aligned}
&(\Theta \otimes 1)(\Theta^{'})(1 \otimes \Theta)\\
&= \sum_{n,i}(\Theta \otimes 1)(\Theta^{'}_i)(1 \otimes \Theta_{n-i})\\
&= \sum_{n}(\Theta \otimes 1)({}^{\tau}\Delta \otimes 1)(\Theta_n)\\
&= \sum_{n}(\Delta \otimes 1)(\Theta_n)(\Theta \otimes 1)\\
&= \sum_{n,i}(1 \otimes \Theta_{n-i})(\Theta^{''}_i)(\Theta \otimes 1)\\
&= (1 \otimes \Theta)(\Theta^{''})(\Theta \otimes 1)
\end{aligned}$$

Where the third equality follows from the definition of Θ and the co-product. But this expression composed with $\tilde{f}_{1,3}\tilde{f}_{1,2}\tilde{f}_{2,3}$ is precisely the right hand side; as is easily seen by using the above relations (and noting that the $\tilde{f}$'s all commute).

6. The hexagon Diagrams

THEOREM 6.1. *The following diagrams commute:*

$$\begin{array}{ccccccc}
M \otimes (M' \otimes M'') & \xrightarrow{1\otimes R} & M \otimes (M'' \otimes M') & \xrightarrow{can} & (M \otimes M'') \otimes M' & \xrightarrow{R\otimes 1} & (M'' \otimes M) \otimes M' \\
{\scriptstyle =}\downarrow & & & & & & {\scriptstyle =}\downarrow \\
M \otimes (M' \otimes M'') & \xrightarrow{can} & (M \otimes M') \otimes M'' & \xrightarrow{R} & M'' \otimes (M \otimes M') & \xrightarrow{can} & (M'' \otimes M) \otimes M'
\end{array}$$

and also

$$\begin{array}{ccccccc}
(M \otimes M') \otimes M'' & \xrightarrow{R\otimes 1} & (M' \otimes M) \otimes M'' & \xrightarrow{can} & M' \otimes (M \otimes M'') & \xrightarrow{1\otimes R} & M' \otimes (M'' \otimes M) \\
{\scriptstyle =}\downarrow & & & & & & {\scriptstyle =}\downarrow \\
(M \otimes M') \otimes M'' & \xrightarrow{can} & M \otimes (M' \otimes M'') & \xrightarrow{R} & (M' \otimes M'') \otimes M & \xrightarrow{can} & M' \otimes (M'' \otimes M)
\end{array}$$

PROOF. We shall prove the bottom diagram, the proof of the top is almost the same. In the top part of this diagram, the first R is $\Theta_{1,2}\tilde{f}_{1,2}P_{1,2}$ while the second R is $\Theta_{2,3}\tilde{f}_{2,3}P_{2,3}$. Therefore, we consider the composition, which is

$$\begin{aligned}&\Theta_{2,3}\tilde{f}_{2,3}P_{2,3}\Theta_{1,2}\tilde{f}_{1,2}P_{1,2}\\ &= \Theta_{2,3}\tilde{f}_{2,3}\Theta_{1,3}P_{2,3}\tilde{f}_{1,2}P_{1,2}\\ &= \Theta_{2,3}\tilde{f}_{2,3}\Theta_{1,3}\tilde{f}_{1,3}P_{2,3}P_{1,2}\\ &= \Theta_{2,3}\Theta^{''}\tilde{f}_{2,3}\tilde{f}_{1,3}P_{2,3}P_{1,2}\end{aligned}$$

where we have used the following equalities: $P_{2,3}\Theta_{1,2} = \Theta_{1,3}P_{2,3}$ and $P_{2,3}\tilde{f}_{1,2} = \tilde{f}_{1,3}P_{2,3}$ and $\tilde{f}_{2,3}\Theta_{1,3} = \Theta^{''}\tilde{f}_{2,3}$. The last one was proved in the previous section, while the first two are immediate consequences of the definitions. Now, the lower half of the diagram involves only one R, which is given by the permutation $(132) = (23)(12)$, followed by the diagonal matrix $\tilde{f}(\lambda\mu, \nu)$ (on a weight vector in $M_\lambda \otimes M_\mu \otimes M_\nu$), followed by $(\Delta \otimes 1)(\Theta)$ (as the action on the tensor product is defined by Δ). But we also have $(\Delta \otimes 1)(\Theta) = (\Theta_{2,3})(\Theta^{''})$, so combining this with the relation $\tilde{f}(\lambda\mu, \nu) = \tilde{f}(\lambda, \nu)\tilde{f}(\mu, \nu)$ shows that the two halves of the diagram are equal. $\square$

CHAPTER 6

Geometric Representation Theory for $SL_q(2)$

1. The Quantum Coordinate Algebra of SL_2

In this section, we provide two independent constructions of a Hopf algebra $O_q(SL_2)$, which plays the role of the coordinate algebra $O(SL_2)$ in the quantum setting. First, we construct $O_q(SL_2)$ as the algebra of matrix coefficients associated to $U_q(\mathfrak{sl}_2)$, as in Chapter 2. Secondly, we introduce a simple non-commutative algebra called the quantum plane, construct an algebra $O_q(Mat_2)$, the universal bi-algebra co-acting on the quantum plane, and inside there exhibit a central "q-determinant", which we may set to one, to obtain $O_q(SL_2)$. (haven't written this up yet...)

2. Peter-Weyl style definition of O_q

Definition 2.1. The quantized coordinate algebra, $O_q(SL(2))$, henceforth denoted O_q, is the subspace of U_q^* spanned by matrix coefficients of type I representations, i.e., linear functionals of the form $c_{f,v}(u) := f(uv)$, for V a type I representation of U_q, $f \in V^*, v \in V$.

That O_q is a subalgebra follows immediately from the formula $c_{f,e}c_{f',e'} = c_{f\otimes f',e\otimes e'}$ (as in the classical case). We give O_q a coalgebra structure via $\Delta(c_{f_i,e_j}) = \sum_k c_{f_i,e_k} \otimes c_{f_k,e_j}$. An antipode is obtained from that on U_q, via the bi-linear pairing between U_q and O_q: for $a \in O_q$, we define $S(a)$ by the formula:

$$\langle S(a), x\rangle = \langle a, S(x)\rangle,$$

where $x \in U_q$ is arbitrary. That this makes O_q into a Hopf algebra is an easy verification, just as in the classical case.

Inspecting the proof of the Peter-Weyl Theorem for classical SL_2, we see that the proof hinged only on the fact that the category of $U_q(\mathfrak{sl}_2)$-modules is semi-simple, and that we had an explicit list of all simple objects. The category of finite-dimensional $U_q(\mathfrak{sl}_2)$-modules is also semi-simple, and its simple objects are in bijection with those of $U(\mathfrak{sl}_2)$. Thus we have:

Proposition 2.2. *There exists an isomorphism of* $U_q(\mathfrak{sl}_2)\otimes U_q(\mathfrak{sl}_2)$*-modules,*

$$O_q \cong \bigoplus_k V^*(k) \boxtimes V(k).$$

Remark 2.3. There is one subtlety in the construction of O_q: by design the algebra $O(SL_2)$ was equivariant for $U(\mathfrak{sl}_2) \otimes U(\mathfrak{sl}_2)$. We should expect the same for O_q, that it be equivariant for the action of $U_q \otimes U_q$. However, there is a catch, which is that the antipode $S : U_q \to U_q$ is an anti-automorphism of the coproduct, i.e. $\Delta(S(x)) =$

$S(x_{(2)}) \otimes S(x_{(1)})$. This means that O_q is natural an algebra in the category $\mathcal{C}^{op} \boxtimes \mathcal{C}$, not $\mathcal{C} \boxtimes \mathcal{C}$. This is merely a reflection of contravariance of the duality functor $* : \mathcal{C} \to \mathcal{C}$.

We wish to derive a "generators and relations" presentation of O_q, from its definition as matrix coefficients. To begin, we note that, as before, the module $V(1)$ generates all finite dimensional representations in the sense that $V(n) \subset V(1)^{\otimes n}$ (as follows from Clebsch-Gordan).

Letting $\mathbb{C}\langle a, b, c, d\rangle$ denote the free algebra on symbols a, b, c, d, we have a surjection,

$$\mathbb{C}\langle a, b, c, d\rangle \twoheadrightarrow O_q,$$
$$(a, b, c, d) \mapsto (c_{f^0,v_0}, c_{f^1,v_0}, c_{f^0,v_1}, c_{f^1,v_1}).$$

Now, we can use the R-matrix to compute the commutativity relations between the matrix coefficients of $V(1)$, which we label a, b, c, d, where $a = c_{0,0}$, $b = c_{0,1}$, $c = c_{1,0}$ and $d = c_{1,1}$. We label the R-matrix entries $R_{i,j}^{k,l}$ and these are given by

$$c_{V,V}(v_i \otimes v_j) = \sum R_{i,j}^{k,l} v_l \otimes v_k.$$

Then we recall from the previous lecture that we have

$$R_{0,0}^{0,0} = R_{1,1}^{1,1} = q^{-1}, \quad R_{0,1}^{1,0} = R_{1,0}^{0,1} = 1, \quad R_{1,0}^{1,0} = q - q^{-1},$$

and all remaining entries zero. These coefficients imply the following

LEMMA 2.4. *The generators a, b, c, d satisfy the following relations:*

$$ab = qba, \quad bc = cb, \quad cd = qdc, \quad ac = qca,$$
$$bd = qdb, \quad ad - da = (q - q^{-1})bc, \quad ad - qbc = 1.$$

PROOF. Each of the purely quadratic relations is obtained by applying the relations,

$$\sum R_{kl}^{ij} a_m^k a_n^l = \sum R_{kl}^{ij} c_{f^l \otimes f^k, v_m \otimes v_n} = c_{\sigma^*(f^i \otimes f^j), v_m \otimes v_n}$$
$$= c_{f^i \otimes f^j, \sigma(v_m \otimes v_n)} = \sum c_{f^i \otimes f^j, v_s \otimes v_t} R_{mn}^{ts} = \sum a_s^j a_t^i R_{mn}^{ts},$$

the so-called Fadeev-Reshetikhin-Takhtajian (FRT) relations.

For instance,

$$\begin{aligned} qba &= qc_{0,1}c_{0,0} = qc_{f^0 \otimes f^0, v_1 \otimes v_0} = c_{\sigma^*(f^0 \otimes f^0), v_1 \otimes v_0} \\ &= c_{f^0 \otimes f^0, c(v^1 \otimes v_0)} = c_{f^0 \otimes f^0, v_0 \otimes v_1} = c_{0,0}c_{0,1} = ab \\ ad &= c_{0,0}c_{1,1} = c_{f^1 \otimes f^0, v_0 \otimes v_1} = c_{\sigma^*(f^0 \otimes f^1), v_0 \otimes v_1} \\ &= c_{f^0 \otimes f^1, \sigma(v_0 \otimes v_1)} = c_{f_0 \otimes f_1, v_1 \otimes v_0} + (q - q^{-1}) c_{f^0 \otimes f_1, v^0 \otimes v_1} \\ &= c_{1,1}c_{0,0} + (q - q^{-1}) c_{1,0}c_{0,1} = da + (q - q^{-1})cb. \end{aligned}$$

The remaining quadratic relations are proved similarly. The determinant relation follows, as in the classical SL_2 computation. □

Exercise 2.5. Using the PBW theorem for $O_q(SL_2)$, show that the above generators and relations yield a presentation of O_q. (Hint: all the ingredients of Exercise ??) can be applied mutatis mutandis to the quantum setting.

The comultiplication on $O_q(SL_2)$ is given by the same formulas as in classical $O(SL_2)$:

$$\begin{pmatrix} \Delta(a) & \Delta(b) \\ \Delta(c) & \Delta(d) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
$$= \begin{pmatrix} a\otimes a + b\otimes c & a\otimes b + b\otimes d \\ c\otimes a + d\otimes c & c\otimes b + d\otimes d \end{pmatrix}.$$

The antipode is given by:

$$\begin{pmatrix} S(a) & S(b) \\ S(c) & S(d) \end{pmatrix} = \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix}.$$

One checks easily that this is an antipode on $SL_q(2)$; by uniqueness, it coincides with the antipode given by the pairing with U_q.

3. O_q comodules

Let M be a right O_q-comodule. Then we can put a left U_q-module structure on M as follows: by definition there is a map $\Delta : M \to M \otimes O_q$. Therefore we have maps $U_q \otimes M \to U_q \otimes M \otimes O_q \to M \otimes U_q \otimes O_q \to M$ where the second to last map is the flip and the last is $1\otimes <,>$. By the properties of the Hopf pairing, this map makes M into a left U module. In particular, this association is a functor from right O_q comodules to left U_q modules, which, when restricted to finite dimensional M, yields only type 1 U_q modules. This is because the weights of K on M are given by coefficients of eigenvectors coming from expressions of the form $< K, o >$ for $o \in O_q$; but the collection of these is $\{q^n\}_{n\in\mathbb{Z}}$ as O_q is defined using only type 1 modules. Our remaining aim in this lecture is to show

Theorem 3.1. *The functor from finite dimensional right O_q comodules to type 1 finite dimensional left U_q modules is an equivalence of categories.*

Proof. In general, suppose C is a coalgebra and M a finite dimensional comodule. Let $\{m_1, ..., m_n\}$ be a basis for M. Then we can write the coaction as $\Delta m_i = \sum_j m_j \otimes c_{j,i}$. Now, coassociativity of this action tells us that it is the same to apply $\Delta_M \otimes 1$ and $1 \otimes \Delta_C$. The first gives

$\sum_{j,k} m_k \otimes c_{k,j} \otimes c_{j,i}$, and so this implies that $\Delta c_{i,j} = \sum_j c_{k,j} \otimes c_{j,i}$, or, in matrix notation, $\Delta(c_{r,m}) = (c_{r,m}) \otimes (c_{r,m})$. Further, from the counit axiom, $\epsilon c_{i,j} = \delta_{i,j}$. Now, if you are given a collection of n^2 elements of C called $(c_{i,j})$, whose counit and comultiplication satisfy the above relations, then clearly the same formula $\Delta m_i = \sum_j m_j \otimes c_{j,i}$ makes M into a C-comodule. Thus, if $C = U_q$ and M is a finite dimensional type 1 module, then the matrix coefficients for this module satisfy these relations by definition. So in fact we have a natural right O_q comodule structure on M, as required. □

4. The Borel, torus, and unipotent radical for $O_q(SL_2)$.

In the quantum case, we don't have the groups or Lie algebras per se; what we have is their quantum enveloping algebras $\mathfrak{U}_\mathfrak{q}$ and the corresponding matrix coefficients $\mathcal{O}_q$. As above, we consider $G = SL(2)$, and define the following subalgebras of $U_q(\mathfrak{sl}_2)$:

$$\begin{aligned} U_q(\mathfrak{b}) &= \mathbb{C} < E, K, K^{-1} > / < KEK^{-1} = q^2 E >, \\ U_q(\mathfrak{t}) &= \mathbb{C}[K, K^{-1}]. \\ U_q(\mathfrak{n}) &= \mathbb{C}[E] \end{aligned}$$

As before, we can check that the first two define Hopf subalgebras, i.e. that they are closed with respect to co-products and antipodes defined on $U_q(\mathfrak{sl}_2)$. We have inclusions $U_q(\mathfrak{t}) \subset U_q(\mathfrak{b}) \subset U_q(\mathfrak{g})$. We can also define an algebra $U_q(\mathfrak{n}) = \mathbb{C}[E]$. However, this isn't a Hopf algebra, because $\Delta(E) = E \otimes 1 + K \otimes E$. What we do have is that $\Delta(U_q(\mathfrak{n})) \subset U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{n})$. Thus, if V is a $U_q(\mathfrak{n})$-module, W a $U_q(\mathfrak{sl}_2)$-module, we can still define $V \otimes W$ a $U_q(\mathfrak{n})$-module, by $E(m \otimes n) = Em \otimes n + Km \otimes En$.

On the level of algebras of functions, we have maps,

$$\mathcal{O}_q(G) \twoheadrightarrow \mathcal{O}_q(B) = \mathcal{O}_q(G)/\langle c \rangle \twoheadrightarrow \mathcal{O}_q(T) = \mathcal{O}_q(B)/\langle b \rangle.$$

As above, one checks that the defining ideals are in fact Hopf ideals, so that these are Hopf algebras. One can define a co-algebra $\mathcal{O}_q(N)$ dual to $U_q(\mathfrak{n})$, but it will not have an algebra structure, only a co-algebra structure.

Something interesting happens when we look at $\mathcal{O}_q(T)$. All of the q-commutation relations drop out, so that there is an isomorphism of Hopf algebras $\mathcal{O}_q(T) \cong \mathcal{O}(T)$. Similarly, these two have equivalent abelian categories of comodules, which are just $\mathbb{Z}$-graded vector spaces $M = \oplus M_n$, where $M_n = \{v | \Delta(v) = v \otimes a^n\}$. However, as braided tensor categories they are distinct, because in $\mathcal{O}_q(T)$, when you braid

$M_n \otimes M_m \to M_m \otimes M_n$, you get a factor of $q^{\frac{mn}{2}}$ that is not there in the classical case.

4.1. Quantum $\mathbb{P}^1$ as flag variety of SL_2. Recall that in the classical case, the induction functor had an interpretation as the global sections of B-equivariant bundles on the flag variety (which for SL(2) is just $\mathbb{P}^1$). How should we define quantum $\mathbb{P}^1_q$ so as to generalize this interpretation of the induction functor? As it turns out, we won't be able to make sense of $\mathbb{P}^1_q$ as a space in its own right. Instead, we will just pretend that it makes sense as an algebraic variety, and proceed to define its quasi-coherent sheaves, via analogy. This approach is somewhat justified due to the fact that in the classical case, invertible sheaves are of the form $\mathcal{O}(n) = \mathcal{O}(1)^{\otimes n}$, and we can recover $\mathbb{P}^1$ by taking $\mathrm{Proj}(\Gamma_*(\mathcal{O}(1))$ (see Hartshorne, p. 117-119 for these constructions). Thus, in the classical case, the category of quasi-coherent sheaves contains a subcategory of invertible sheaves, which, taken altogether can be used to recover $\mathbb{P}^1$ itself.

In analogy with the situation of affine algebraic groups, we'd like to define quasi-coherent sheaves on $\mathbb{P}^1_q$ as B_q-equivariant $\mathcal{O}_q$-modules (here $\mathcal{O}_q$ means the structure sheaf on the "group variety" G_q); sadly B_q and G_q don't exist as actual varieties either; only their algebras of functions make sense. So we'll have to take a different perspective.

Definition 4.1. $\mathcal{QC}oh(\mathbb{P}^1_q)$ is the category whose objects are $\mathcal{O}_q(SL_2)$-modules M, which are also $\mathcal{O}_q(B)$-comodules, such that the module map $\mathcal{O}_q \otimes M \to M$ is an $\mathcal{O}_q(B)$ co-module map, where $\mathcal{O}_q \otimes M$ has the tensor product co-module structure. Morphisms are maps that are compatible with both actions.

$\mathcal{O}_q(B)$ co-modules are morally just B_q-modules (which are not defined), and this is the motivation for the definition, so that for $q = 1$, this gives back the category of modules on the flag variety $\mathbb{P}^1 = SL(2)/B$.

Example 4.2. O_q itself with the restricted co-module action gives a quasi-coherent sheaf on $\mathbb{P}^1_q$.

Example 4.3. For any V a $O_q(B)$-comodule, $O_q \otimes V$ gives another quasi-coherent sheaf on $\mathbb{P}^1_q$, where the new co-module product is that induced by the tensor product (not just the original action on V).

Example 4.4. $O_q \otimes \mathbb{C}_n = \mathcal{O}_q(n)$, is the twisting sheaf on $\mathbb{P}^1_q$.

Definition 4.5. If $M \in \mathcal{QC}oh(\mathbb{P}^1_q)$, we define $\Gamma(M) = \mathrm{Hom}_{\mathbb{P}^1_q}(\mathcal{O}_q, M)$.

Lemma 4.6. $\Gamma(M) = M^{B_q}$.

PROOF. Compatibility with the $\mathcal{O}_q$ structure would give M, corresponding to where the identity element is to be sent (as per the usual isomorphism $\mathrm{Hom}_A(A, M) \cong M$, for M and A-module). Compatbility with the comodule structure implies that the identity must be sent to an invariant element. $\square$

The following Borel-Weil theorem has the same proof as in the classical case:

THEOREM 4.7. *(Borel-Weil)* $\Gamma(\mathcal{O}_q(n)) \cong V(n)^*$.

DEFINITION 4.8. ${}_{SL_q(2)}qCoh(\mathbb{P}^1_q)$ is the category whose objects are $\mathcal{O}_q(SL_2)$-modules, which are also right $\mathcal{O}_q(B)$ co-modules and left $\mathcal{O}_q(SL_2)$ co-modules, and so that the module map $\mathcal{O}_q \otimes M \to M$ is both an $\mathcal{O}_q(B)$ and $\mathcal{O}_q(SL_2)$ co-module map. Morphisms in this category are those which commute with all the actions.

This is a somewhat cumbersome definition. Fortunately, it is equivalent to a much more reasonable category.

LEMMA 4.9. ${}_{SL_q(2)}qCoh(\mathbb{P}^1_q) \cong \mathcal{O}_q(B)$*-comod.*

PROOF. If V is an $\mathcal{O}_q(B)$-comodule, then we can send V to $\mathcal{O}_q \otimes V$. On the other hand, given $M \in_{SL_q(2)} qCoh(\mathbb{P}^1_q)$, we can take ${}^{\mathcal{O}_q(G)}M$, which will be an $\mathcal{O}_q(B)$-comodule. $\square$

When $q = 1$, we have (at least) two ways of constructing $\mathbb{P}^1$. One is as the flag variety of SL_2, as described above, while the other is as the variety, $\mathbb{P}^1 = (\mathbb{A}^2 \backslash \{0\})/\mathbb{C}^\times$. We want to generalize this second construction to quantum $\mathbb{P}^1_q$.

DEFINITION 4.10. $\mathcal{QC}oh(\widetilde{\mathbb{P}^1_q})$ is the category of graded modules over $\mathbb{C} < x, y > / < xy - qyx >$, modulo the category of torsion modules (i.e. $\forall m \in M, \exists i >> 0$ s.t. $x^i m = y^i m = 0$).

In the next few lectures, we will show that in fact the constructions $\mathbb{P}^1_q$ and $\widetilde{\mathbb{P}^1_q}$ are equivalent. The construction of $\mathbb{P}^1_q$ may be used to define $\mathbb{P}^n_q$ as graded modules over $\mathbb{C} < x_0, \ldots, x_n > / < x_i y_j - q_{ij} y_j x_i >$, modulo torsion. More generally, given any graded algebra A, we can define $Proj(A)$ to the the category of A modules, modulo torsion.

4.2. An equivalence of categories arising from the Hopf pairing.

DEFINITION 4.11. M is *integrable* if it splits into a (possibly infinite) direct sum of type I irreducible modules $V(n)$.

REMARK 4.12. Equivalently, A type-I $U_q(\mathfrak{b})$-module M is integrable if we can write $M = \oplus_n M_n$, where $M_n = \{m | Km = q^n m\}$, and $\dim(U_q(\mathfrak{b})m) < \infty, \forall m$.

We have a Hopf pairing ϕ between $\mathcal{O}_q(B)$ and $U_q(\mathfrak{b})$, because we constructed $\mathcal{O}_q(G)$ as a subset of $U_q(\mathfrak{b})^*$ of matrix coefficients. Thus, given an $\mathcal{O}_q(B)$-comodule V, we may define a $U_q(\mathfrak{b})$-module structure on V by

$$U_q(\mathfrak{b}) \otimes V \to^{id \otimes \Delta} U_q(\mathfrak{b}) \otimes V \otimes \mathcal{O}_q(B) \to^{\text{swap}} U_q(\mathfrak{b}) \otimes \mathcal{O}_q(B) \otimes V \to^{\phi \otimes id} \mathbb{C} \otimes V \cong V.$$

LEMMA 4.13. *The above construction satisfies the associativity and unit axiom, and thus induces an equivalence of categories F : (right)$\mathcal{O}_q(B)$-comodules $\to$ (left) integrable $U_q(\mathfrak{b})$-modules..*

PROOF. First, we check that the unit, $1 \in U_q(\mathfrak{b})$, acts as the identity on V.

$$1 \otimes x \mapsto \sum_{(x)} \phi(1 \otimes x_{\mathcal{O}}) x_V = \sum_{(v)} \epsilon(x_{\mathcal{O}}) x_V = x,$$

by the co-unit axiom for V as a $\mathcal{O}_q(B)$ co-module. And we check associativity:

$$\begin{aligned} ab \otimes x \quad &\mapsto \quad \sum_{(x)} \phi(ab \otimes x_{\mathcal{O}}) x_V = \sum_{(x)} \phi(x'_{\mathcal{O}}(a) x''_{\mathcal{O}}(b)) x_V \\ &= \quad \sum_{(x)} \phi(x_{\mathcal{O}}(a) x_{V\mathcal{O}}(b)) x_{VV} = \mu(a \otimes (bx)). \end{aligned}$$

The summation notation used is Sweedler's notation, from e.g. Kassel's *Quantum Groups.*

That you get integrable modules in this way is essentially clear: an $\mathcal{O}_q(B)$-comodule M is already split into weight spaces by the $\mathcal{O}_q(T)$ action: $M = \oplus_n M_n$. By duality, each M_n will be a type-I weight space, of weight n. The local finite condition follows from the analogous property for co-modules over a co-algebra. It remains to show that F is essentially surjective. We have already shown that F hits all finite dimensional U_q modules. Then, since integrable modules are direct sums of these, it is easy to see that F hits all of these too.

4.3. Restriction and Induction Functors.

DEFINITION 4.14. Define the restriction functor,

$$\text{Res}_B^G : \mathcal{O}_q(G) - \text{comod} \to \mathcal{O}_q(B) - \text{comod},$$

with the same underlying vector space, and co-action given by:

$$M \mapsto \mathcal{O}_q(G) \otimes M \twoheadrightarrow \mathcal{O}_q(B) \otimes M.$$

DEFINITION 4.15. Define the induction functor,

$$\mathrm{Ind}_B^G : \mathcal{O}_q(B) - \text{comod} \to \mathcal{O}_q(G) - \text{comod}$$
$$Ind_B^G : M \mapsto (\mathcal{O}_q(G) \otimes M)^{\mathcal{O}_q(B)}$$

Here, $V^{\mathcal{O}_q(B)} = \{v \in V | \ \Delta(m) = m \otimes 1\}$. We use the fact that $\mathcal{O}_q(G)$ has two commuting $\mathcal{O}_q(G)$-comodule structures, coming from left multiplication and right-inverse multiplication. We take the invariants with respect to (say) the right-inverse multiplication (which kills that action and the action on M), and thus have an induced left comodule structure on the invariants coming from the left multiplication.

PROPOSITION 4.16. *(Ind_B^G, Res_B^G) is an adjoint pair.*

PROOF. Given $\phi : (\mathcal{O}_q(G) \otimes M)^{\mathcal{O}_q(B)} \to N$, we construct $\psi = \phi|_{M \otimes 1} : M \to N$ (a quick check verifies that $1 \otimes M$ is invariant, so ϕ is defined there). This defines the adjunction in one direction. For the other direction, given $\psi : M \to N$, we define $\phi : (\mathcal{O}_q(G) \otimes M)^{\mathcal{O}_q(B)} \to (\mathcal{O}_q(B) \otimes M)^{\mathcal{O}_q(B)} \cong M \to N$. These transformations, being mutually inverse, give the desired isomorphism. □

Now we want to consider what a 1-dimensional $\mathcal{O}_q(B)$-comodule would look like. Later we will apply the induction functor to such modules to recover the representations $V(n)^*$.

DEFINITION 4.17. An element χ in a Hopf algebra is called group-like if $\Delta(\chi) = \chi \otimes \chi$.

Now let M be a 1-dimensional $\mathcal{O}_q(B)$-comodule, with basis m. $\Delta(m) = m \otimes a$, for some $a \in \mathcal{O}_q(B)$. Applying co-associativity, we see that a must be group-like. Inside $\mathcal{O}_q(B)$, the only group like elements are of the form $a^n, n \in \mathbb{Z}$. So let us define $\mathbb{C}_m$ to be the 1-dimensional co-module with basis 1_m, s.t. $\Delta(1_m) = 1_m \otimes a^{-n}$.

THEOREM 4.18. *$Ind_B^G(\mathbb{C}_m) = V(m)^*$.*

PROOF. By the Peter-Weyl Theorem,

$$\mathcal{O}_q(SL(2)) = \bigoplus_{n \in \mathbb{Z}} V(n)^* \otimes V(n).$$

Thus, tensoring with $\mathbb{C}_m$ and taking invariants, we get,

$$[\mathcal{O}_q(SL(2)) \otimes \mathbb{C}_n]^{\mathcal{O}_q(B)} = [\bigoplus_{n \in \mathbb{Z}} V(n)^* \otimes V(n) \otimes \mathbb{C}_m]^{\mathcal{O}_q(B)}$$

Since taking $\mathcal{O}_q(B)$-invariants picks out the zeroeth graded component with respect to the $\mathcal{O}_q(B)$ action, and since the gradings on the tensor add, we pick out the component corresonding to $n = m$ (Since we chose

$\mathbb{C}_m$ to be of weight $-m$). This trivializes the action on the two right components, and so all we are left with is the left action on $V(n)^*$. □

5. Lecture 14 - Quasi-coherent sheaves

5.1. Classical case. Let us recall the basic example of $G = SL_2$, for which we have $N, T \subset B \subset G$ as previously defined. In this case, we have $G/B \cong \mathbb{P}^1$. Indeed, B is a semi-direct product $T \ltimes N$, and G acts transitively on $\mathbb{A}^2 - \{0\}$, with stabilizer N, hence we get:

$$G/B \cong (G/N)/T \cong \frac{\mathbb{A}^2 - \{0\}}{T} \cong \frac{\mathbb{A}^2 - \{0\}}{\mathbb{C}^*} = \mathbb{P}^1.$$

Our goal is to find an analog of this in the quantum case. We would like to have objects $N_q, T_q \subset B_q \subset G_q = SL_{2,q}$ satisfying the following:

$$G_q/N_q \cong \mathbb{A}^2_q - \{0\}, G_q/B_q \cong \mathbb{P}^1_q.$$

However, as we have seen previously, these objects don't exist, only their algebras of functions do. This is why we turn our attention to quasi-coherent sheaves, which in the classical case allow us to recover the spaces. In this setup we have the category $qCoh(SL_2/N)$ of $\mathcal{O}(SL_2)$-modules which are also N-modules in a compatible way, i.e. the map $\mathcal{O}(SL_2) \otimes M \to M$ is a map of N-modules.

We have seen previously that as categories, the following equivalences hold.

$$qCoh(\mathbb{A}^2) \cong \mathbb{C}[x, y]\text{-modules},$$
$$qCoh(\mathbb{A}^2 - \{0\}) \cong \mathbb{C}[x, y]\text{-modules/torsion modules},$$

and the restriction functor $qCoh(\mathbb{A}^2) \twoheadrightarrow qCoh(\mathbb{A}^2 - \{0\})$ corresponds to the quotient functor. In fact, the map $i : \mathbb{A}^2 - \{0\} \hookrightarrow \mathbb{A}^2$ induces a pair of adjoint functors (i^*, i_*). We will construct an analog of this in our new language, without reference to actual spaces.

Lemma 5.1. $\mathcal{O}(SL_2)^N \cong \mathbb{C}[x, y]$.

Proof. Recall the following fact:

$$\mathcal{O}(SL_2) = \mathbb{C}[c_{f_0,e_0}, c_{f_1,e_0}, c_{f_0,e_1}, c_{f_1,e_1}]/(det = 1),$$

where c_{f_i,e_j} are the usual matrix coefficients. There are two actions of SL_2 on $\mathcal{O}(SL_2)$, given by $g \cdot c_{f,v} = c_{f,gv}$ or $c_{gf,v}$. Taking, say, the first one, we see that c_{f_0,e_0} and c_{f_1,e_0} are N-invariant, and in fact, N-invariants cannot have terms involving c_{f_0,e_1} or c_{f_1,e_1}. Thus we get $\mathcal{O}(SL_2)^N = \mathbb{C}[c_{f_0,e_0}, c_{f_1,e_0}]$. Were we to take the second action instead, we would obtain $\mathcal{O}(SL_2)^N = \mathbb{C}[c_{f_1,e_0}, c_{f_1,e_1}]$. In any case, the claim holds. □

REMARK 5.2. Another way to prove this would be to use the Peter-Weyl theorem: $\mathcal{O}(SL_2) = \mathrm{op}V(n)^* \otimes V(n)$. Computing N-invariants, we obtain:

$$\mathcal{O}(SL_2)^N = \mathrm{op}V(n)^*.$$

As an algebra over $\mathbb{C}$, this is generated (freely) by f_0, f_1, the dual basis of $V(1)^*$.

Using the lemma, we can define the following functor F.

$$\mathbb{C}[x,y]\text{-modules} \underset{G}{\overset{F}{\rightleftarrows}} qCoh(SL_2/N)$$

$$F : M \mapsto \mathcal{O}(SL_2) \underset{\mathcal{O}(SL_2)^N}{\otimes} M$$

$$G : M \mapsto M^N.$$

Let us check that FM is indeed an object of $qCoh(SL_2/N)$. It is clearly an $\mathcal{O}(SL_2)$-module, and inherits the structure of N-module from $\mathcal{O}(SL_2)$, via $n \cdot (f \otimes m) = (n \cdot f) \otimes m$. To see that this is well defined, let $f \in \mathcal{O}(SL_2), \alpha \in \mathcal{O}(SL_2)^N$, and $m \in M$.

$$\begin{aligned} n \cdot (\alpha f \otimes m) &= n \cdot (\alpha f) \otimes m \\ &= (n \cdot \alpha)(n \cdot f) \otimes m \\ &= \alpha (n \cdot f) \otimes m \quad \text{(since } \alpha \text{ is } N\text{-invariant)} \\ &= (n \cdot f) \otimes \alpha m \\ &= n \cdot (f \otimes \alpha m). \end{aligned}$$

Moreover, the action $\mu : \mathcal{O}(SL_2) \otimes FM \to FM$ is a map of N-modules.

$$\begin{aligned} \mu\left(n \cdot (f \otimes f' \otimes m)\right) &= \mu\left(n \cdot f \otimes n \cdot (f' \otimes m)\right) \\ &= \mu\left(n \cdot f \otimes n \cdot f' \otimes m\right) \\ &= (n \cdot f)(n \cdot f') \otimes m \\ &= (n \cdot f f') \otimes m \\ &= n \cdot (f f' \otimes m) \\ &= n \cdot \mu(f \otimes f' \otimes m). \end{aligned}$$

PROPOSITION 5.3. *F is a quotient by torsion modules, i.e.*
(a) F is unto;
(b) As a subcategory, $F^{-1}(0)$ is the category of torsion modules.

PROOF. (a) For any object M in $qCoh(SL_2/N)$, we have $FGM \cong M$. Indeed, this is true for the structure sheaf $\mathcal{O}(SL_2)$.

$$FG\left(\mathcal{O}(SL_2)\right) = \mathcal{O}(SL_2) \underset{\mathcal{O}(SL_2)^N}{\otimes} \mathcal{O}(SL_2)^N \cong \mathcal{O}(SL_2).$$

And the category is generated by its structure sheaf, hence the result holds for any M.

(b) Recall the following:

$$SL_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| ad - bc = 1 \right\};$$
$$\mathcal{O}(SL_2) = \mathbb{C}[a, b, c, d]/(ad - bc = 1);$$
$$\mathcal{O}(SL_2)^N = \mathbb{C}[a, c].$$

Let M be a torsion module, i.e. one on which a, c act locally nilpotently. We want to show that $FM = \mathcal{O}(SL_2) \otimes_{\mathcal{O}(SL_2)^N} M$ is zero.

Since a and d commute, ad acts locally nilpotently on FM, and similarly for bc. Thus $ad - bc$ acts locally nilpotently on FM, but we know it is 1, hence it acts as the identity. Therefore every element of FM is zero.

Conversely, assume FM is zero, and take $m \in M$. Since $1 \otimes m$ is zero in FM, we must have, for some ks large enough, $(ad - bc)^k \otimes m = 0$ "on the nose", i.e. in $\mathbb{C}[a, b, c, d] \otimes_{\mathcal{O}(SL_2)^N} M$. Expanding this and using commutation, we obtain:

$$\sum_{i=0}^{k} \kappa_i a^i b^{k-i} c^{k-i} d^i \otimes m = \sum_{i=0}^{k} b^{k-i} d^i \otimes \kappa_i a^i c^{k-i} m = 0.$$

Each term of this sum must be zero, and thus each right-hand factor is zero in M. In particular, $a^k m$ and $c^k m$ are zero.
Therefore, FM is zero iff a, c act locally nilpotently on M. □

5.2. $\mathbb{G}_m$-equivariant construction of quantum $\mathbb{P}^1$. In analogy to what we have done, we define $qCoh(G_q/B_q)$ as the category of left $\mathcal{O}_q(SL_2)$-modules which are also right $\mathcal{O}_q(B)$-comodules such that the module structure $\mathcal{O}_q(SL_2) \otimes M \to M$ is a map of $\mathcal{O}_q(B)$-comodules.

THEOREM 5.4. *$qCoh(G_q/B_q) \cong \mathrm{Proj}\,(\mathbb{C} < x, y > /xy = qyx)$.*

This is the category of $\mathbb{Z}$-graded modules over $\mathbb{C} < x, y > /(xy = qyx)$ modulo torsion modules, i.e. those on which x, y act locally nilpotently. For the grading we have $deg(x) = deg(y) = 1$.

Before proving this, we define in a similar way $qCoh(G_q/N_q)$ as the category of left $\mathcal{O}_q(SL_2)$-modules which are also right $\mathcal{O}_q(N)$-comodules such that $\mathcal{O}_q(SL_2) \otimes M \to M$ is a map of $\mathcal{O}_q(N)$-comodules.

REMARK 5.5. $\mathcal{O}_q(N)$ is NOT a Hopf algebra, as in the classical case.

Let us recall what are the objects we are working with.
We have $\mathcal{O}_q(SL_2) \twoheadrightarrow \mathcal{O}_q(B) \twoheadrightarrow \mathcal{O}_b(T), \mathcal{O}_q(N)$, where:

$$\mathcal{O}_q(SL_2) \cong \mathbb{C} < a, b, c, d > /\text{following relations}$$

$$ad - qbc = 1, ab = qba, cd = qdc, ac = qca, bc = cb, bd = qdb,$$
$$ad - da = (q - q^{-1})bc.$$

$$\mathcal{O}_q(B) \cong \mathcal{O}_q(SL_2)/ < c >$$

$$\mathcal{O}_q(T) \cong \mathcal{O}_q(B)/ < b >$$

$$\mathcal{O}_q(N) \cong \mathcal{O}_q(B)/\mathcal{O}_q(B)(a - 1).$$

Here $< b >$ and $< c >$ denote the Hopf ideals generated by b and c respectively. Note that $\mathcal{O}_q(SL_2), \mathcal{O}_q(B), \mathcal{O}_q(T)$ are Hopf algebras, and $\mathcal{O}_q(N)$ is a coalgebra but fails to be an algebra. This is due to the fact that the quantum enveloping algebra $U_q(N) = \mathbb{C}[E]$ fails to be a coalgebra, since it is not closed under coproduct. Indeed, $E \in U_q(sl_2)$ satisfies $\Delta E = E \otimes 1 + K \otimes E$.

Now to prove the theorem, we need to prove the following fact. Let us denote $\mathbb{A}^2_q := \mathbb{C} < x, y > /(xy = qyx)$, called the **quantum plane**.

PROPOSITION 5.6. *$qCoh(G_q/N_q) \cong$ category of modules over $\mathbb{A}^2_q$ modulo torsion modules.*

PROOF. The proof of proposition (5.3) essentially works in this case also. We use the same argument to show that FM is zero iff x and y act locally nilpotently on it. The commutation relations for $\mathcal{O}_q(SL_2)$ make the computations messier, but the result still holds. ☐

To complete the proof of the theorem, notice that an object of $qCoh(G_q/B_q)$ is like an object of $qCoh(G_q/N_q)$ with an extra structure of $\mathcal{O}_q(T)$-comodule. However, we know that $\mathcal{O}_q(T)$ is equal to $\mathcal{O}(T)$, namely $\mathbb{C}[a, a^{-1}]$. Hence an $\mathcal{O}_q(T)$-comodule structure is an $\mathcal{O}(T)$-comodule structure, which is equivalent to a T-module structure. Here T is just a 1-dimensional torus, so this torus action corresponds to a $\mathbb{Z}$-grading.

5.3. Quantum differential operators on $\mathbb{A}_q^2$. Recall that the differential operators on $\mathbb{A}^2$ are given by the 2^{nd} Weyl algebra.

$$\mathrm{Diff}(\mathbb{A}^2) = W_2 = \mathbb{C} < x, y, \partial_x, \partial_y > /\text{following relations}$$

$$[x, y] = [\partial_x, \partial_y] = [\partial_x, y] = [\partial_y, x] = 0; [\partial_x, x] = [\partial_y, y] = 1.$$

To define a quantum analog, we could try the following naive deformation.

$$W_{2,q} = \mathbb{C} < x, y, \partial_x, \partial_y > /\text{following relations}$$

$$xy = qyx,\ \partial_x\partial_y = q^{-1}\partial_y\partial_x,\ \partial_x x - qx\partial_x = 1,\ \partial_y y - qy\partial_y = 1.$$

The problem is that $U_q(sl_2)$ does NOT embed in this $W_{2,q}$, so this is not the deformation we are looking for. Instead we will use another approach to differential operators.

5.3.1. *Differential operators à la Grothendieck.* Starting with a commutative $\mathbb{C}$-algebra A, we define $\mathrm{Diff}(A) \subset End_{\mathbb{C}}(A)$ through a filtration $\mathrm{Diff}_0(A) \subset \mathrm{Diff}_1(A) \subset \cdots$, setting $\mathrm{Diff}(A) = \bigcup_n \mathrm{Diff}_n(A)$.

$$\mathrm{Diff}^0(A) = A$$

$$\mathrm{Diff}^{n+1}(A) = \{\varphi \in End_{\mathbb{C}}(A) \mid [\varphi, a] \in \mathrm{Diff}^n(A)\ \forall a \in A\}$$

Here we view $a \in A$ as the endomorphism l_a of left-multiplication by a.

EXAMPLE 5.7. $\mathrm{Diff}(\mathbb{C}[x_1, \ldots, x_n]) = W_n$, the n^{th} Weyl algebra, defined as:

$$\mathbb{C} < x_1, \ldots, x_n, \partial_1, \ldots, \partial_n > /[\partial_i, x_i] = 1 \text{ and all other generators commute.}$$

Writing $A = \mathbb{C}[x_1, \ldots, x_n]$, we first compute $\mathrm{Diff}^1(A)$.

Notice that $\varphi \in End_{\mathbb{C}}(A)$ is a derivation iff it satisfies $\varphi(1) = 0$ and $\varphi \in \mathrm{Diff}_1(A)$. Indeed, a derivation clearly satisfies $\varphi(1) = 0$, and among such endomorphisms, the condition of being in $\mathrm{Diff}^1(A)$ becomes:

$$\begin{aligned} & \varphi l_a - l_a\varphi = l_b \text{ for some } b = l_b(1) \\ \Leftrightarrow\quad & \varphi l_a - l_a\varphi = l_{\varphi(a)} \text{ since } \varphi(1) = 0 \\ \Leftrightarrow\quad & \varphi(ax) - a\varphi(x) = \varphi(a)x, \end{aligned}$$

for all $x, a \in A$, that is, φ is a derivation. Thus we have the short exact sequence:

$$0 \longrightarrow \mathrm{Der}(A) \longrightarrow \mathrm{Diff}^1(A) \xrightarrow{ev_1} A \longrightarrow 0$$

which splits, for example via the embedding $l : A \hookrightarrow \mathrm{Diff}^1(A)$ of left-multiplication. Thus we have:

$$\mathrm{Diff}_1(A) \cong A\mathrm{op}\mathrm{Der}(A).$$

We know that $\partial_1, \ldots, \partial_n$ are derivations, but in fact, any derivation $d \in \mathrm{Der}(A)$ is generated by these over A. Namely, we have:

$$d = \sum_{i=1}^{n} d(x_i)\partial_i.$$

An inductive step allows us to show that $\mathrm{Diff}^n(A)$ as a left A-module is generated (freely) by all monomials in $\partial_1, \ldots, \partial_n$ of degree at most n. Taking the union over all n, we obtain the algebra W_n. Indeed, the algebra structure is the free structure with the given commutation relations as only relations.

Remark 5.8. The algebra of differential operators over a singular variety can be much more complicated than this.

5.3.2. *Generalization to the quantum case.* We want to use this definition of differential operators to define quantum differential operators over the quantum plane, i.e. on the algebra:

$$A_q := \mathbb{C} < x, y > /(xy = qyx).$$

We need to be careful, as the naive application of the definition will not yield what we are looking for. Instead, let us use the fact that A_q is a $U_q(sl_2)$-module algebra, i.e. the multiplication map $\mu : A_q \otimes A_q \to A_q$ is a map of $U_q(sl_2)$-modules. Moreover, it is commutative with respect to the R-matrix, i.e. the following diagram commutes.

$$\begin{array}{ccc} A_q \otimes A_q & \xrightarrow{\mu} & A_q \\ \downarrow R & \nearrow \mu & \\ A_q \otimes A_q & & \end{array}$$

Recall that $\mathcal{O}_q(SL_2)$ is commutative in the category of $\mathcal{O}_q \otimes \mathcal{O}_q^{co-op}$-comodules, with respect to the R-matrix on $U_q(sl_2)$, which becomes $R \otimes R^{-1}$. If we take the invariants $\mathcal{O}_q^{N_q} \subset \mathcal{O}_q$, it is still commutative with respect to $R \otimes R^{-1}$. Furthermore, R is of the form:

$$\sum_n a_n F_n \otimes E_n \circ \tilde{f}.$$

Thus when we apply $R \otimes R^{-1}$ to $\mathcal{O}_q^{N_q}$, the milpotent part of R^{-1} acts trivially (only the identity survives), and we are left with $R \otimes \tilde{f}$.

Define the category of $\mathbb{Z}$-graded $U_q(sl_2)$-modules with an R-matrix of the form $R \circ \tilde{f} \otimes (q^{\frac{1}{2}})^{deg(a)deg(b)}$. Note that $U_q(sl_2)$ is already graded by weight, and here we consider an additional grading.

CLAIM 5.9. *$A_q \simeq opV_n^*$ is a commutative algebra in this category of graded $U_q(sl_2)$-modules.*

Define $\underline{End}(A_q) \subset End_{\mathbb{C}}(A_q)$ as all sums of homogeneous endomorphisms (with respect to both gradings). Another way to define this is by looking at:

$$U_q(sl_2) \otimes \mathbb{C}[T, T^{-1}].$$

Both factors are Hopf algebras, hence so is their tensor product. $End_{\mathbb{C}}(A_q)$ is also a module over this algebra, and $\underline{End}(A_q)$ consists of the endomorphisms that are semisimple with respect to K and T.

Our next goal will be to define a commutator:

$$[\ ,\ ]_n : \underline{End}(A_q) \otimes A_q \to \underline{End}(A_q)$$

and use it to define quantum differential operators $\mathrm{Diff}_q(A_q)$. We will then compute these, and see that $U_q(sl_2)$ and $\mathrm{Diff}_q^0(A_q)$ are closely related, although not equal in general.

6. Quantum D-modules

In this lecture, our goal is to define quantum differential operators. In the classical case, we defined differential operators inductively; for an algebra A, we defined

$$D_{k+1}(A) = \{\phi \in \mathrm{End}(A) \mid [\phi, L_a] \in D_k(A) \quad (\forall a \in A)\}$$

where L_a denotes left multiplication by A. We'll give a similar definition for the quantum case. However, since tensor products are not commutative but are R-commutative, we will need to define an R-commutator.

To obtain the closest parallels to the classical case, we will need to limit which algebras A we consider. Of course we'll only consider integrable modules, but we need another condition too. We'll do this by introducing an operator T, which we should think of as the quantum version of the Euler operator $x\partial_x + y\partial_y$. We'll only consider algebras A on which T acts similarly to the Euler operator in the classical case; this is the condition we need so that everything works out nicely.

Recall that $U_q(\mathrm{SL}(2))$ has center $\mathbb{C}[C]$, where C is the Casimir

$$C = FE + \frac{Kq + K^{-1}q^{-1}}{(q - q^{-1})^2}$$

Just as in the classical case, the Casimir "separates irreducibles", in the sense that C acts on $V(m)$ by multiplication by $\frac{q^{m+1}+q^{-m-1}}{(q-q^{-1})^2}$. We introduce a new formal parameter T, and define a map $\mathbb{C}[C] \to \mathbb{C}[T, T^{-1}]$

taking $C \mapsto \frac{Tq+T^{-1}q^{-1}}{(q-q^{-1})^2}$. Of course, T is not in U_q, and in general it will not be possible to define an action of T on a U_q-module (in a way agreeing with the action of C). However, we can extend the action to T for irreducible modules $V(m)$: T simply acts by multiplication by q^m. Thus, we see that T represents the quantum version of the classical Euler operator $x\partial_x + y\partial_y$ (which also acts by multiplication on irreducibles).

We want to limit ourselves to the category of algebras which interact nicely with T. To express this condition, we define the extended algebra

$$\tilde{U}_q = U_q \otimes_{\mathbb{C}(C)} \mathbb{C}[T, T^{-1}]$$

We only want to consider integrable U_q-algebras which have a $\tilde{U}_q$-module structure. Another way of saying this is that we want to consider U_q-modules with a $\mathbb{Z}$-grading corresponding to highest weights (that is, a vector v is graded by the highest weight of the irreducible subrepresentation containing it).

We also need to consider how the R matrix behaves with respect to the T-grading. Suppose M is in an integrable U_q-representation. As usual, we let M_n denote the nth graded piece $M_n = \{m \in M | Km = q^n m\}$. Recall that for $v \in M_{n'}, w \in M_{m'}$, we used the function Θ_{-K} defined as

$$\Theta_{-K}(v \otimes w) = q^{-m'n'/2}(v \otimes w)$$

Then our R matrix is $R = \sum a_n F^n \otimes E^n \circ \Theta_{-K}$. We want to shift the emphasis from the weights to our new T-grading by highest weights instead; so, we define Θ_T as follows. Suppose that v and w are contained in irreducible subrepresentations $V(n)$ and $V(m)$ respectively. Then

$$\Theta_T(v \otimes w) = q^{mn/2}(v \otimes w)$$

We define a new R matrix which also accounts for the T-grading: $\tilde{R} = R \circ \Theta_T$.

Let's look at our fundamental example $U_q(\mathrm{SL}(2))$. We want to define the quantum differential operators on $\mathbb{A}^2_q$. Comparing to the classical case, we expect that we should examine endomorphisms of

$$\mathcal{O}_q^{N_q} = \mathbb{C} < x, y > / \, (xy = qyx)$$

For ease of notation, we'll denote this algebra by A_q.

CLAIM 6.1. (1) *$A_q = \oplus V^*(n)$ is a $\mathbb{Z}$-graded integrable U_q-module.*

(2) *A_q is in fact a U_q-module algebra, that is, the multiplication map $A_q \otimes A_q \to A_q$ is a map of U_q-modules.*

(3) *A_q is commutative with respect to $\tilde{R}$ (up to powers of q). Writing $R = (R_0 \otimes R_1) \circ \Theta_{-K}$ in our usual summation notation, this means that $ab = q^c R_0(b) R_1(a)$ for some appropriate power q^c depending on a and b.*

We already proved 1 and 2, and 3 follows from our R-matrix computations earlier.

Now we analyze $\mathrm{End}_{\mathbb{C}}(A_q)$. There is an adjoint action of U_q on $\mathrm{End}_{\mathbb{C}}(A_q)$: $u(f)(a) = u_1 f(S u_2 \cdot a)$.

When defining differentials, we shouldn't allow every endomorphism; we need to limit ourselves to endomorphisms that work well with the T-grading if we want to mimic the classical situation. Thus, we take our differentials from the inner endomorphisms of A_q in the category of $\mathbb{Z}$-graded integrable U_q-modules. These endomorphisms don't necessarily preserve the grading, but they only change it "finitely". That is, we should be able to write the endomorphism as a finite sum of its graded pieces. We denote this subring by $\underline{\mathrm{End}}(A_q)$.

We also need to define a quantum commutator that respects the gradings. Define the auxiliary function $\epsilon_i : A_q \to \mathbb{C}$ so that it takes $v \in V(n)$ to $\epsilon_i(v) = q^{2in}$. Also, let $m : \underline{\mathrm{End}}(A_q) \otimes A_q \to \underline{\mathrm{End}}(A_q)$ denote the natural multiplication, so $m(f \otimes r) = f \circ L_r$. Then, we define $[,]_i : \underline{\mathrm{End}}(A_q) \otimes A_q \to \underline{\mathrm{End}}(A_q)$ to be

$$[,]_i = m \; - \; m \circ \tilde{R} \circ \mathrm{flip} \circ (\mathrm{Id} \otimes \epsilon_i)$$

To be absolutely clear, we rewrite this action explicitly. For $r \in A_q$ and $f \in \underline{\mathrm{End}}(A_q)$, define $\theta_i = \epsilon_i(r) \Theta_T(L_r, f) \Theta_{-K}(L_r, f)$. So, θ_i accounts for all the factors of q that occur. Then

$$[f, r]_i = f \circ L_r \; - \; \theta_i(r, f) L_{R_0(r)} R_1(f)$$

This changes the degrees in the appropriate way. If we did not use this graded commutator, we would have too few differential operators - we would end up with just left multiplication.

Lemma 6.2. *For all $f, g \in \underline{End}(A_q)$, and $r \in A_q$, we have*

$$[f \circ g, r]_{j+k} = f \circ [g, r]_j + \theta_j(r, g)[f, R_0(r)]_k R_1(g)$$

This lemma follows from the hexagon diagram we discussed earlier. We'll use it to show that differential operators form a ring in the usual way.

Finally, we can define the differential operators inductively. Let $D_{-1}(A_q) = 0$, and define

$$D_{k+1}(A_q) = \{\phi \in \underline{\mathrm{End}}(A_q) \mid [\phi, L_a]_k \in D_k(A_q) \quad (\forall a \in A_q)\}$$

Note that the commutator changes each step, so that it always has the right grading action. As usual, we let $D(A_q)$ be the union of the $D_k(A_q)$. We can start analyzing the differential operators just as in the classical case.

Definition 6.3. Let (n) denote the quantum quantity $(q^{2n}-1)/(q^2-1)$. Define $\mathbb{C}$-linear endomorphisms of A_q:

$$\partial_x(y^n x^m) = q^n (m) y^n x^{m-1}$$
$$\partial_y(y^n x^m) = (n) y^{n-1} x^m$$

The q^n factor arsies from commuting x^m across y^n.

The following lemma shows that quantum differential operators behave just like their classical counterparts.

Lemma 6.4. (1) *$D(A_q)$ is a ring under composition.*
(2) $D_0(A_q) = A_q$
(3) $\partial_x, \partial_y \in D_1(A_q)$
(4) *$D(A_q)$ is a free left A_q-module with basis $\partial_x^m \partial_y^n$.*

The first part is proven by using the lemma above to show that $D(A_q)$ is fixed under composition. The second part is proven using the q-commutativity of A_q under the $\tilde{R}$-matrix. The third part is proven just as in the classical case, by showing $[\partial, L_a] = \partial(a)$ for any $\partial \in D_1$. The fourth part is also proven just as in the classical case by considering the action on A_q. It is not hard to come up with explicit generators and relations for $D(A_q)$ using this lemma. In particular, the following relations are useful to know.

Claim 6.5.

$$x\partial_x = K^{-1}T\left(\frac{KT-1}{q^2-1}\right)$$
$$y\partial_y = \frac{K^{-1}T-1}{q^2-1}$$
$$x\partial_y = K^{-1}TE$$
$$y\partial_x = q^{-1}TF$$

Finally, we want to identify the 0-graded part D_0 of $D(A_q)$ as a subalgebra inside of $\tilde{U}$. Classically, we have that the algebra

$$\mathbb{C} < x\partial_y, y\partial_x, x\partial_x - y\partial_y > \subset \ W$$

is naturally identified with $U(\mathrm{SL}(2))$. If we include the Euler operator T so as to contain every degree 0 operator in the Weyl algebra, we get $U(\mathrm{SL}(2))[T]/(C = 2T^2 + T)$.

Similarly, in the quantum case, we have $D_0 \subset \tilde{U}_q$. We can quotient out by the relation $T = 1$ to find something inside of U_q - by our above calculation, this subalgebra contains elements corresponding to $K^{-1}E$, K^{-1}, F, but not K or E. We can identify this subalgebra precisely as follows. Every Hopf algebra H has an adjoint action on itself: $h(u) = h_1 u S(h_2)$. It's easy to check that H is a module algebra for the adjoint action. For any H, we define the locally finite part of H to be the subalgebra

$$H^{\text{l.f.}} = \{h \in H \mid \dim(H \cdot_{\text{adj}} h) < \infty\}$$

Classically, we have $U(\mathfrak{g})^{\text{l.f.}} = U(\mathfrak{g})$. It turns out that in the quantum case $U_q(\text{sl}(2))^{\text{l.f.}}$ is the subalgebra of U_q corresponding to the elements of $D_0/(T = 1)$.

Bibliography

[1] J. Stasheff, *Homotopy associativity of H-spaces, I, II*, Trans. Amer. Math. Soc. 108 (1963), 275–292, 293–312.