Ptolemy's theorem, cluster algebras from surfaces and a conjecture on friezes.

CIMPA School - UNPSJB 2023

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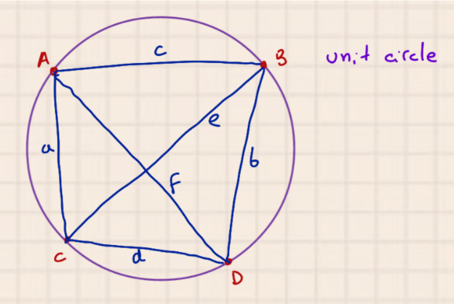
1. Ptolemy's theorem... and friends
2. Marked surfaces and triangulations
3. SNAKES!
4. Friezes from surfaces

Ptolemy's theorem... and friends

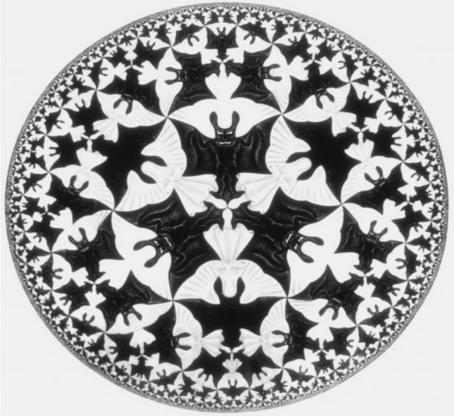
Consider the unit disk and a cyclic euclidean quadrilateral $ABCD$.



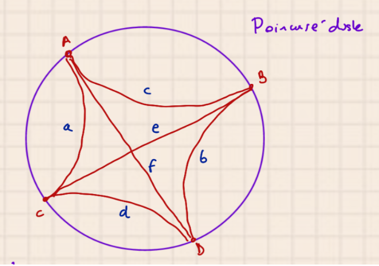
Ptolemy's theorem ($\simeq$ 200 AD) says that

$$ab + cd = ef \qquad * \, ptolemy \;\; relation \, *$$

Consider the Poincaré disk (as a model for the Hyperbolic plane $\mathbb{H}^2$)



Take a quadrilateral $ABCD$ with vertices at $\partial\mathbb{H}^2$.



The edges of the quadrilateral will be 'lines', i.e. *geodesics* in $\mathbb{H}^2$.

We would like to say that the Ptolemy relation holds in this setting...but

DANGER !!!!

In fact the lengths $a, b, c, \ldots$ are infinite here so we need to do some changes.

Penner (1987) introduced the *decorated Teichmüler space* (on marked surfaces)

Take a finite number of *marked points* on $\partial\mathbb{H}^2$. For each point A consider the horocycle h_A (circle of a certain size and tangent to $\partial\mathbb{H}^2$ at A)

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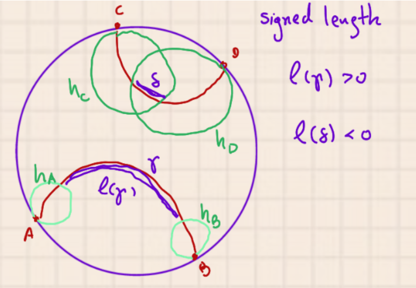
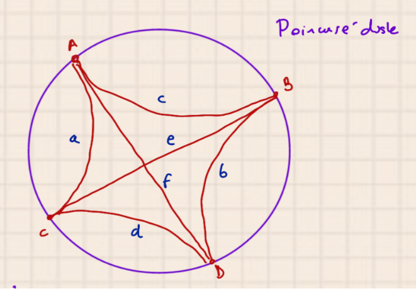


Make the lenghts positive by setting

$$\lambda_a = e^{l(a)/2} \qquad * \textit{ lambda length } *$$

Now we have

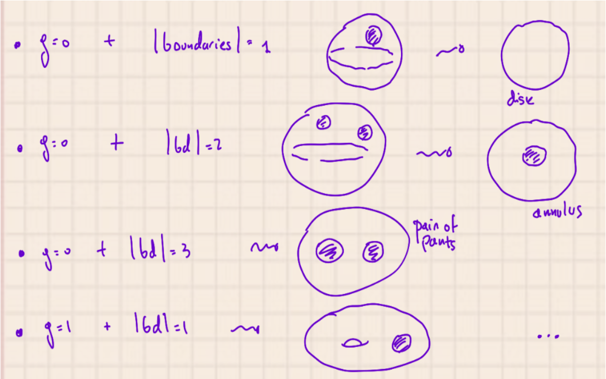


$$\lambda_a \lambda_b + \lambda_c \lambda_d = \lambda_e \lambda_f \qquad * \; \textit{ptolemy relation} \; *$$

Marked surfaces and triangulations

Surface = Riemann, compact, oriented, *with boundary* ∂S.

For that we need two positive parameters g, b (genus and number of boundaries).

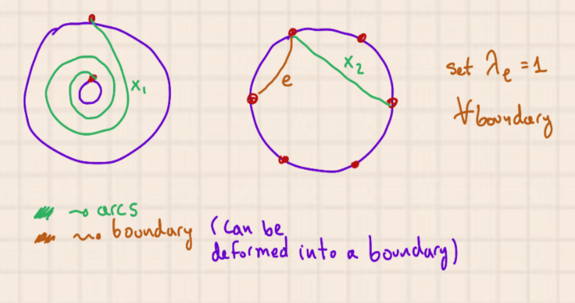


Take a finite set of marked points M on each boundary.

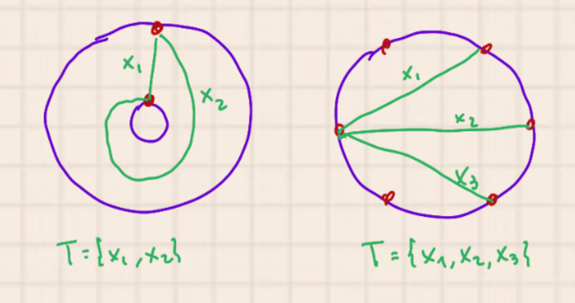
A **marked surface** is a triple $\widetilde{S} = (S, \partial S, M)$.

This marked surface is equipped with a decorated hyperbolic metric with horocycles at each marked point.

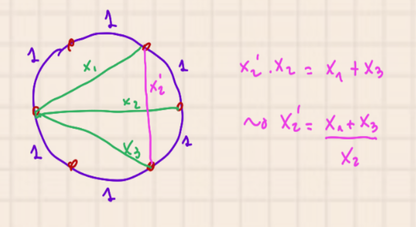
An **arc** is a non self-intersecting curve with endpoints in M. We are interested in lambda lengths for these arcs, which are in time given by certain algebraic expressions.



A **triangulation** T is a set of arcs $\{x_1, \ldots, x_n\}$ splitting the marked surface into triangles.



Let a be an arc, then its lambda length λ_a can be obtained by using Ptolemy relations... recursively.



Definition: (Fomin-Shapiro-Thurston 2008) The **cluster algebra of type** $\widetilde{S}$ is the $\mathbb{Z}$-algebra generated by all the lambda-lengths λ_a for a an arc in $\widetilde{S}$.

For example the algebra of type $\widetilde{S} =$ 'hexagon' is generated by $x_1, x_2, x_3, \dfrac{x_1 + x_3}{x_2}, \ldots$

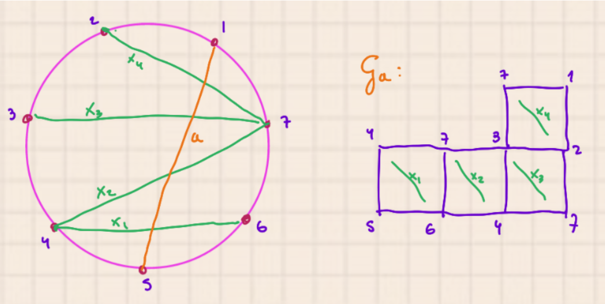
SNAKES!

...using Ptolemy relations... recursively.

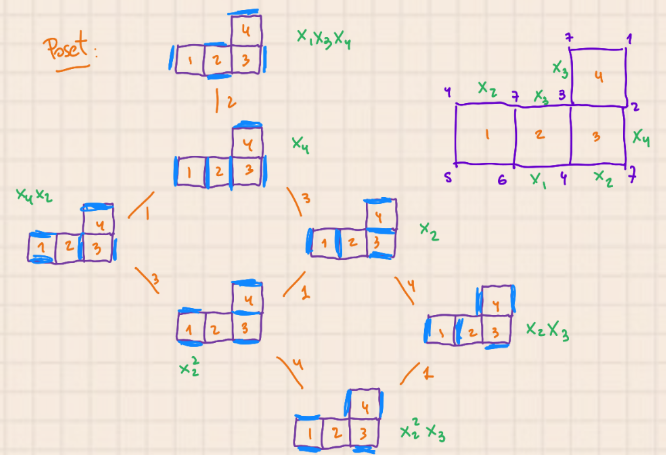
There is a 'compact' formula (Musiker-Schiffler 2010) to compute λ_a in terms of $x_1, \ldots, x_n$.

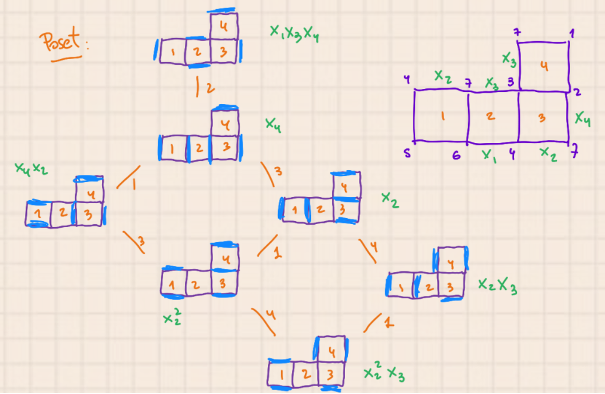
They used a combinatorial tool called *snake graph*.

A *snake graph* $\mathcal{G}$ is a planar graph consisting of a sequence of square faces, where each square is attached either above or to the right of the previous one.



So the expression for λ_a can be computed by looking a the poset from **perfect matchings** of $\mathcal{G}_a$.





It is $\lambda_a = \dfrac{\text{SUM OVER PM}}{\text{CROSS}}$. For our example:

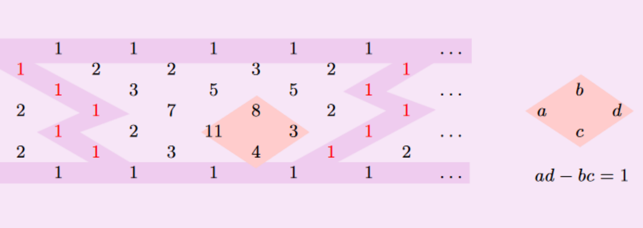
$$\frac{1}{x_1x_2x_3x_4}(x_1x_3x_4 + x_4 + x_2x_4 + x_2 + x_2^2 + x_2x_3 + x_2^2x_3)$$

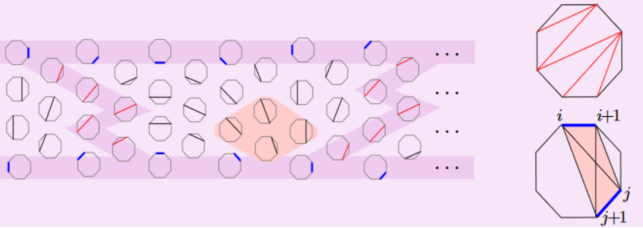
Friezes

Frieze patterns (in the sense of cluster combinatorics) were defined by Conway ('71 - 30 years before cluster theory started!) and studied by **Conway and Coxeter ('73)**.

The **1973 article(s)** is a list of problems to study the properties of frieze patterns, the second paper gives hints and solutions for these problems.

Problems (28) and (29) are known as the **theorem of Conway and Coxeter** and they relate integer friezes with triangulations of polygons.





Figures from Anna Felikson's recent note 'Ptolemy relation and friends' (find it on arxiv!)

(33) Does the 'fundamental region'

$$\begin{array}{ccccccccc}
2 & & 1 & & 4 & & 2 & & 1 \\
 & 1 & & 3 & & 7 & & 1 & \\
 & & 2 & & 5 & & 3 & & \\
 & & & 3 & & 2 & & &
\end{array}$$

(which is repeated upside down, between the upper and lower rows of ones) always contain just $n-3$ ones and the same number of twos?

(34) One simple way to triangulate a polygon is by means of diagonals forming a zigzag. Does this yield a frieze pattern consisting entirely of Fibonacci numbers?

(35) Can we go one step farther and assert that *every frieze pattern of integers either contains a* 4 *or consists entirely of Fibonacci numbers*?

[Section 4 (*Solutions*) will appear in the next issue of the *Gazette*. The references given below apply also to that section.]

References

1. J. Böhm, Zu Coxeters Integrationsmethode in gekrümmten Räumen, *Math. Nachr.*

Now that we have cluster algebras of type $\widetilde{S}$...

A **frieze** $\mathcal{F}$ from $\widetilde{S}$ is

- A map

$$\lambda\colon \{\text{Arcs}\} \to \mathbb{Z}$$

assigning a **positive** number $\lambda(a)$ to every arc a in $\widetilde{S}$ in such a way that the Ptolemy relation holds for every quadrilateral on the surface.

or..

- A ring homomorphism

$$\varphi\colon \mathcal{A}(\widetilde{S}) \to \mathbb{Z}$$

from the cluster algebra to the integers, that is determined by an evaluation $x_i \to a_i \in \mathbb{Z}^+$, such that $\varphi(a) \in \mathbb{Z}^+$ for every cluster variable.

A frieze is **unitary** if there exists a triangulation T of $\widetilde{S}$ such that $\lambda(a) = 1$ for all $a \in T$.

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This is related to problem 33 in the previous slide. The answer to problem 33 is YES.

Given the surface $\widetilde{S}$= disk (at least 4 marked points in ∂S), and a set of lengths $\{\lambda_a \colon a \text{ arc}\}$

- the lengths satisfy Ptolemy relation for all squares
- boundary lengths are all 1
- all λ_a are in $\mathbb{Z}^+$

then (Theorem CC-73 ... or problem 33) there exists a triangulation T of $\widetilde{S}$ such that $\lambda_t = 1$ for all $t \in T$.

Now that we have cluster algebras of type $\widetilde{S}$...

we may ask what happens with other marked surfaces.

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we may ask what happens with other marked surfaces.

Let $\widetilde{S}$ be a marked surface. And let $\mathcal{F}$ be a frieze from $\widetilde{S}$. We know $\mathcal{F}$ is unitary when

- $\widetilde{S}$ is a disk [Conway-Coxeter 73]
- $\widetilde{S}$ is an annulus [Gunawan-Schiffler 2019]
- $\widetilde{S}$ is a pair of pants [Canakci-Felikson-GE-Tumarkin 2021]

Thank you CIMPA Puerto Madryn 2023!

(Emalca 2012)

Some references:

- **Motivation behind cluster algebras:** Fomin Williams Zelevinsky book: Introd. to cluster algebras Chap 1-3, on arxiv.
- **Notes that are similar to this talk but not equal** Anna Felikson's recent note 'Ptolemy relation and friends' on arxiv.
- **For cluster algebras from surfaces** Cluster algebras from surfaces Lecture notes for the CIMPA School Mar del Plata, Schiffler, easy to find online.
- **For an introduction to cluster categories** Cluster algebras and cluster categories Lecture notes for the XVIII Latin American Algebra Colloquium Sao Pedro - Brazil, Schiffler. Easy to find online.
- **To connect cluster categories and friezes** From frieze patterns to cluster categories, Matt Pressland on arxiv.